The Final Merger of Comparable Mass Binary Black Holes

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NASA/GSFC

6th International LISA Symposium
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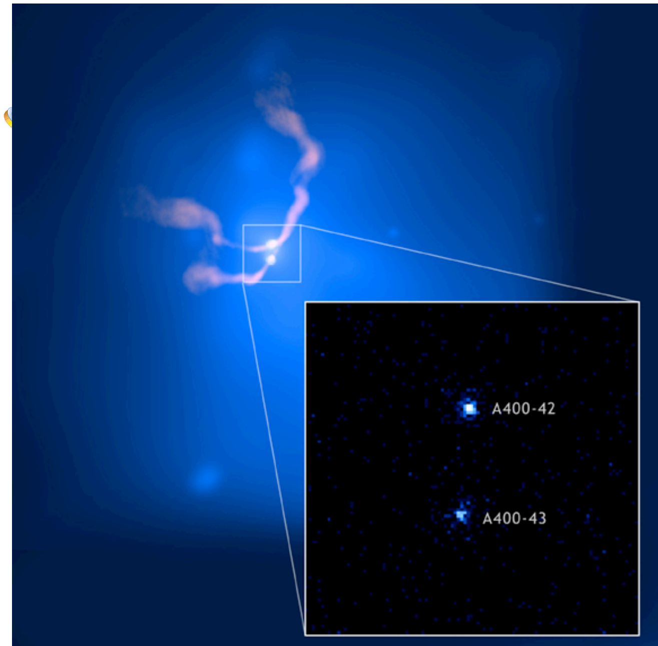
NASA Goddard Space Flight Center

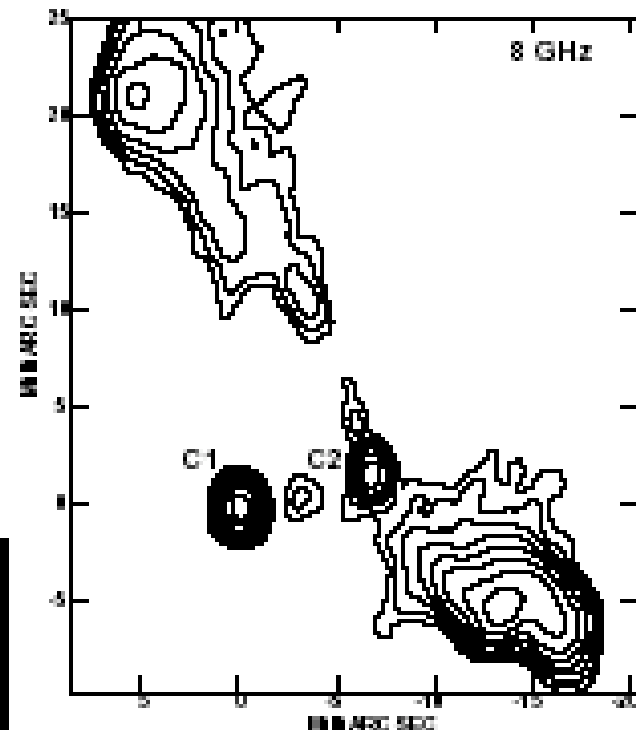
Massive Black Hole binaries...



(X-ray: NASA/CXC/AIfA/D.Hudson & T.Reiprich et all;Radio:NRAO/VLA/NRL)

Abell 400

Separation ~ 7600 pc



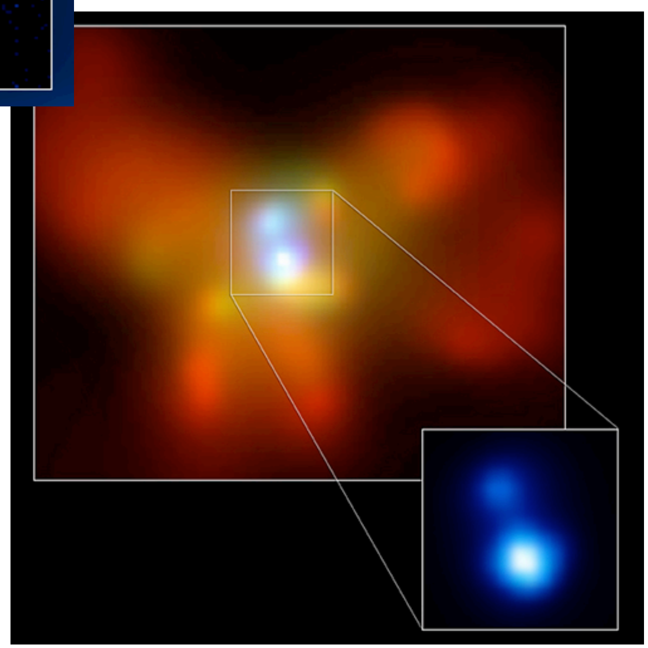
(Rodriguez, et al. ApJ, in press, astro-ph/0604042)

0402+379

Separation ~ 7.3 pc

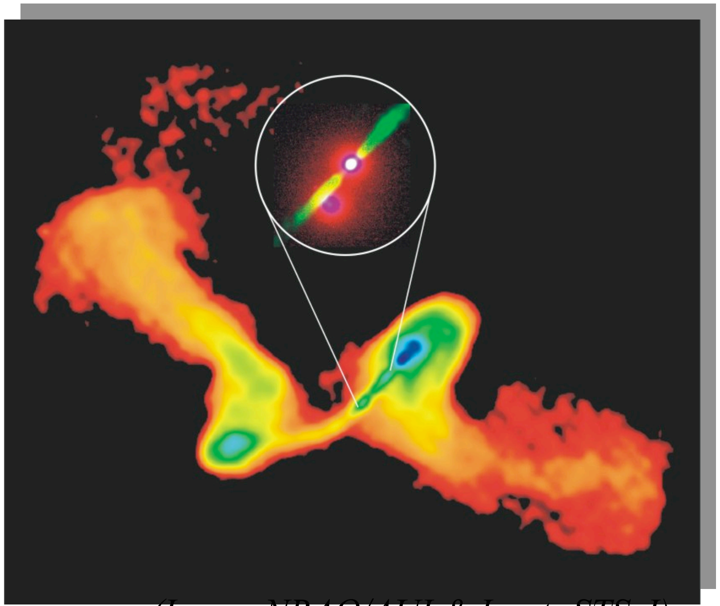
NGC 6240

Separation ~ 1000pc



(NASA/CXC/MPE/S.Komossa et al.)

MBH Mergers & LISA…

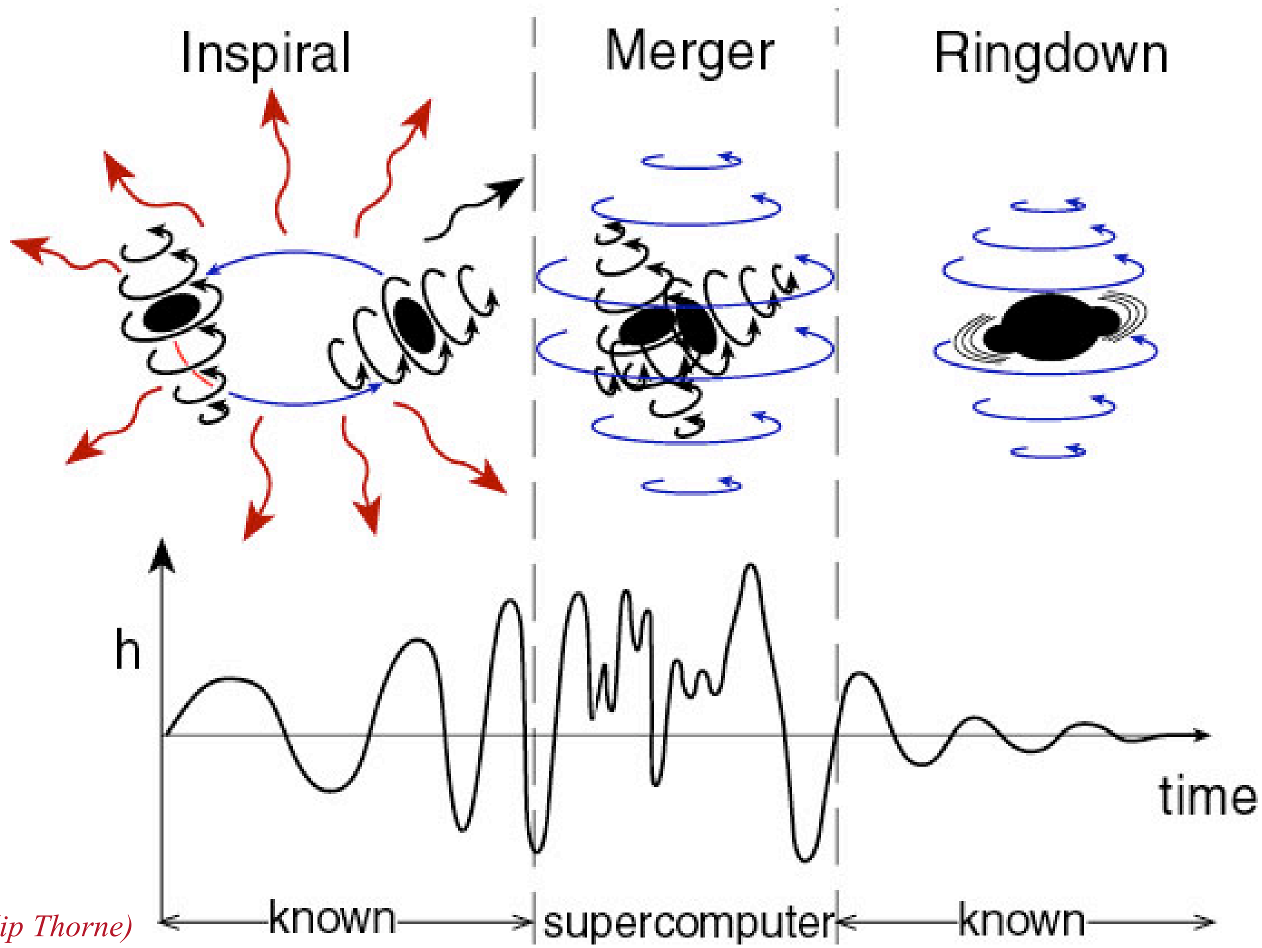


(Image NRAO/AUI & Inset: STScI)

- Gravitational waves encode the dynamics of massive objects
- Observing gravitational waves allows ***direct*** tests of GR
- MBH mergers are very strong LISA sources
- Final merger of MBHs occurs in the arena of extremely strong gravity
- ***LISA can test GR in the dynamical, strong field regime…if we know the merger waveforms***
- When $m_1 \neq m_2$, GW emission is asymmetric ◊ recoil kick
- If this kick is large enough, it could eject the merged remnant from the host structure… and ***affect the rates of merger events***
- MBHs are expected to be spinning…
- MBH mergers could produce ***interesting spin dynamics and couplings***

Final merger of black hole binary

- **Strong-field merger is brightest GW source, luminosity ~ $10^{23} L_{SUN}$**
- **Requires *numerical relativity* to calculate dynamics & waveforms**
- **Waveforms scale w/ masses, spins ◊ apply to ground-based & LISA**



(graphic courtesy of Kip Thorne)

Final merger of black hole binary

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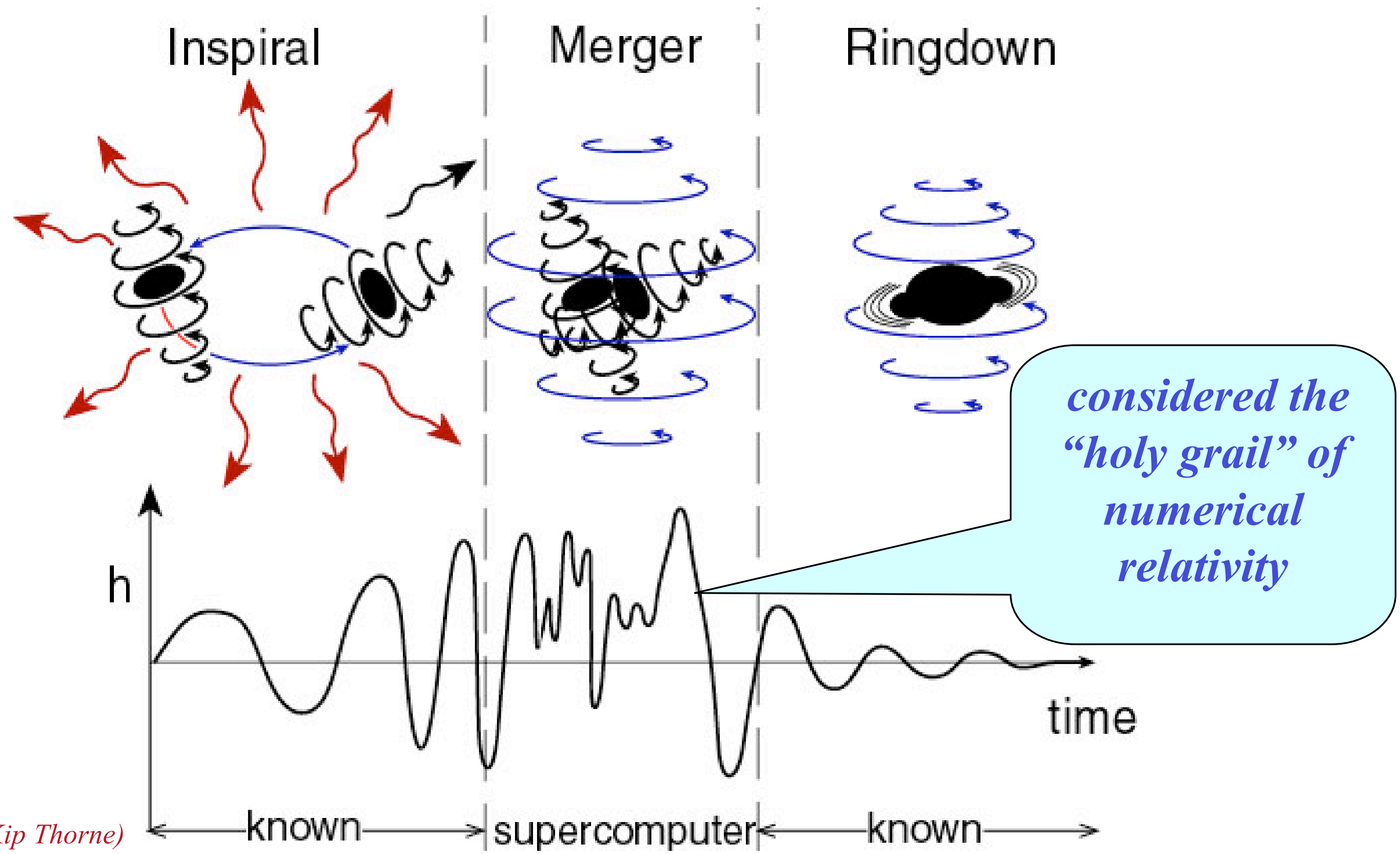


(graphic courtesy of Kip Thorne)

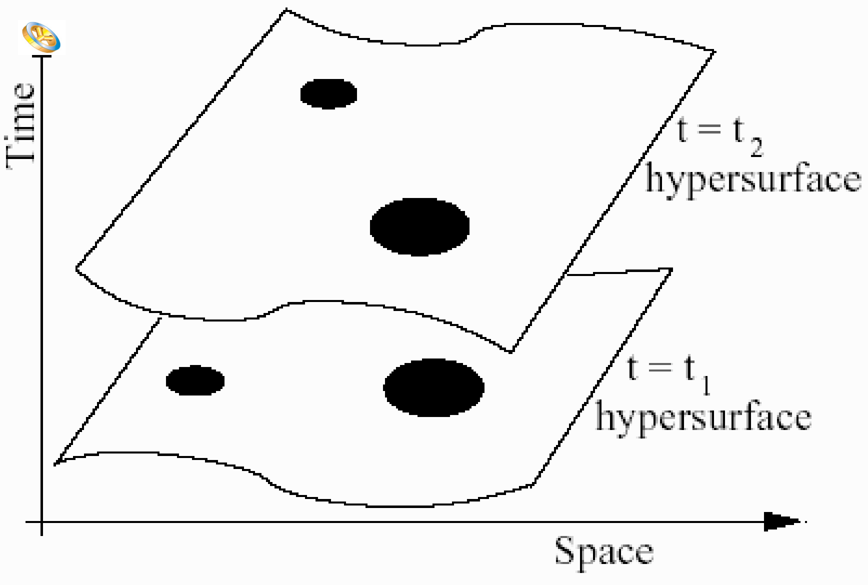
A major challenge….

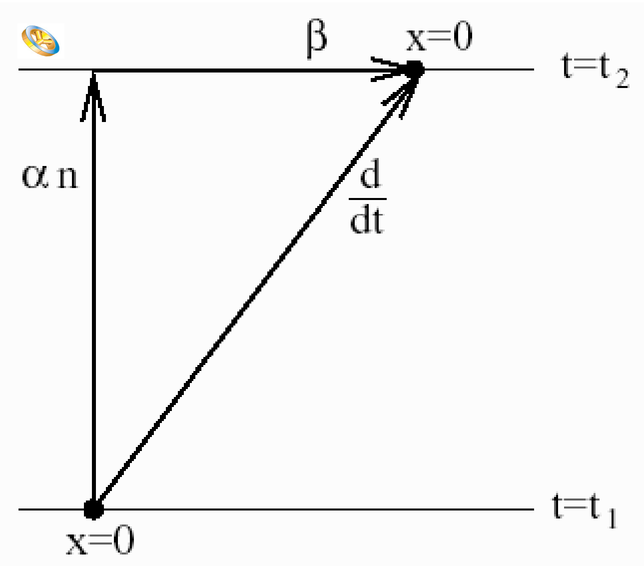
"Nearly as difficult as building these *(gravitational wave)* observatories, however, is the task of **computing the gravitational waveforms** that are expected when two black holes merge. This is a **major challenge** in computational general relativity and one that will stretch computational hardware and software to the limits. However, a bonus is that the **waveforms will be quite unique to general relativity**, and if they are reproduced observationally, scientists will have performed a **highly sensitive test of gravity in the strong-field regime**."

-- excerpt from "What are the Limits of Physical Law?" in ***Connecting Quarks with the Cosmos: Eleven Science Questions for the New Century*** (Board on Physics and Astronomy, National Academies, 2003)

Spacetime Engineering

- **Numerical Relativity….**
- **Solve Einstein eqns on a computer**
- **Spacetime sliced into 3-D**
  t **= constant hypersurfaces**
- **Einstein's eqns split into 2 sets:**
  - **Constraint equations**
  - **Evolution equations**
- **Set (constrained) initial data at t = 0**
- **Evolve forward in time, from one slice to the next**
- **Solve ≥ 17 nonlinear, coupled PDEs**
- **Coordinate or gauge conditions: relate coords on neighboring slices**
  - **lapse function _, shift vector _^{i}**





A Brief History of BBH simulations….

- 1964: Hahn & Lindquist: try to evolve collision of 2 "wormholes"
- 1970s: Smarr and Eppley: head-on collision of 2 BHs, extract GWs
  - Pioneering efforts on supercomputers at Livermore Natl Lab
- 1990s: LIGO moves ahead & work on BBH problem starts up again..
  - Work on 2-D head-on collisions at NCSA
  - NSF Grand Challenge: multi-institution, multi-year effort in 3-D

◊ *This is really difficult! Instabilities, issues in formalisms, etc…*

  - Diaspora: multiple efforts (AEI, UT-Austin, PSU, Cornell…)
  - Difficulties proliferate, instabilities arise, codes crash....
  - *"Numerical relativity is impossible..."*
- 2000 & beyond: LIGO/GEO/VIRGO and LISA spur more development
  - New groups arise: Caltech, UT-Brownsville, LSU, NASA/GSFC…
  - *Since 2004, breakthroughs & rapid progress: orbits, at last!*

Recent progress…on a broad front

- Evolutions of BH binary with equal mass, non-spinning BHs
  - start on approx quasi-circular orbits near last stable orbit
  - stable evolution over multiple orbits, plunge, merger, ringdown
- Independently written codes and different software
  - Finite differences; spectral methods
- Different formulations of the Einstein equations
  - 1^{ST} & 2^{nd} order PDEs; which variables to use; role of constraints
- How to handle the BHs: excision; “punctures”
- Gauge or coordinate conditions: co-moving coordinates; moving BHs
- Variable grid resolution to handle multiple scales: $\_{GW}$ ~ (10 – 100)M
  - Mesh refinement; spectral decomposition
- Units: c = G = 1 ◊ 1 M ~ 5 x 10^{-6} (M/M_{Sun}) sec ~ 1.5 (M/M_{Sun}) km
- Now beginning to study binaries with unequal masses, & with spin….

The 1st complete BBH orbit…

- Bruegmann, Tichy, & Jansen, PRL, 92, 211101 (2004), gr-qc/0312112
- Represent BHs as "punctures":

$$g_{ij} = \psi^4 \delta_{ij} \qquad \psi = \psi_{BL} + u$$

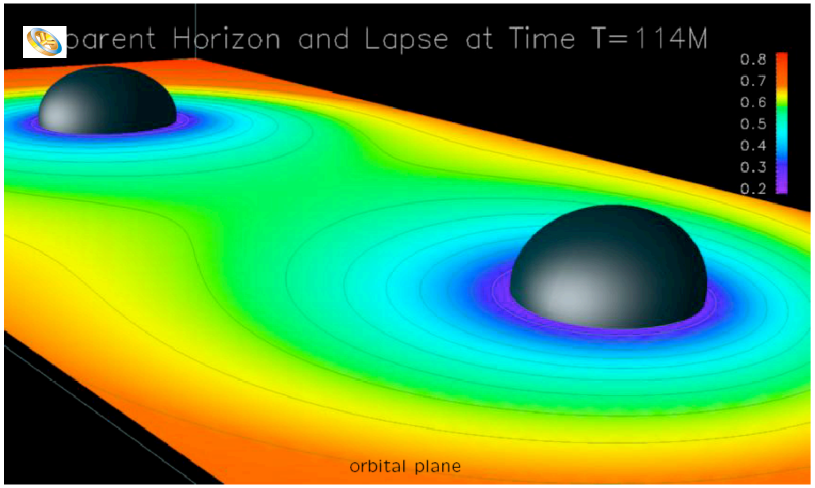
$$\psi_{BL} = 1 + \sum_{n=1}^{2} m_n / 2 | r - r_n |$$

- Handle singular _BL analytically; evolve only nonsingular *u*

◊ fix the BH punctures in the grid

- Use comoving shift vector _
- Conformal formalism
  - g_{ij}, $A_{ij} \sim \partial_t g_{ij}$
  - 1st order time, 2nd order space

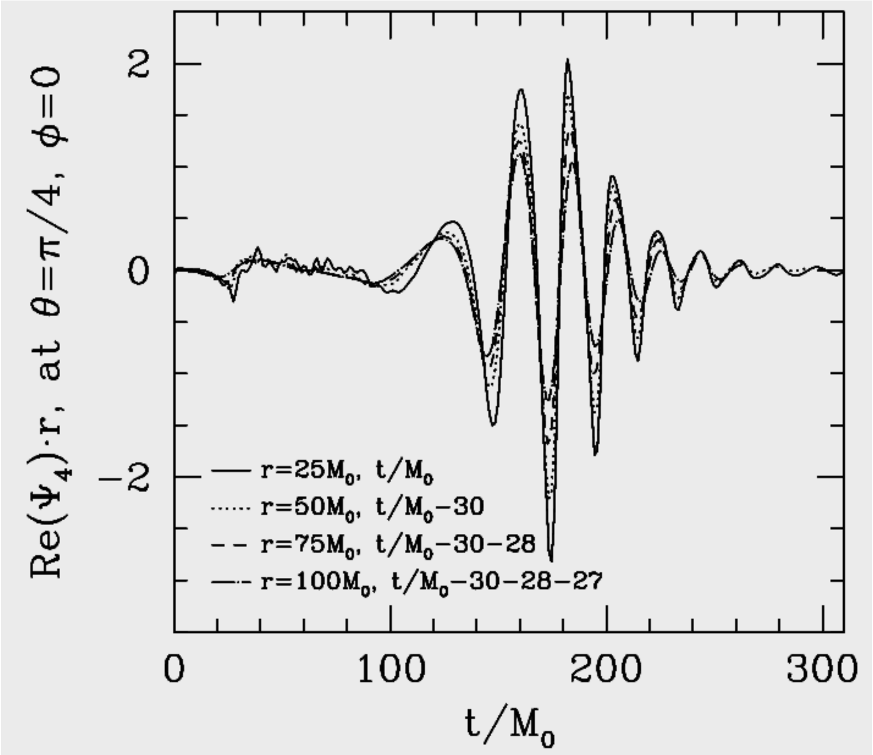
- Excise BHS at late times



- Runs for ~ (125 – 150)*M* and BHs complete ~ 1 orbit
- Crashes before BHs merge
- Not accurate enough to be able to extract GWs

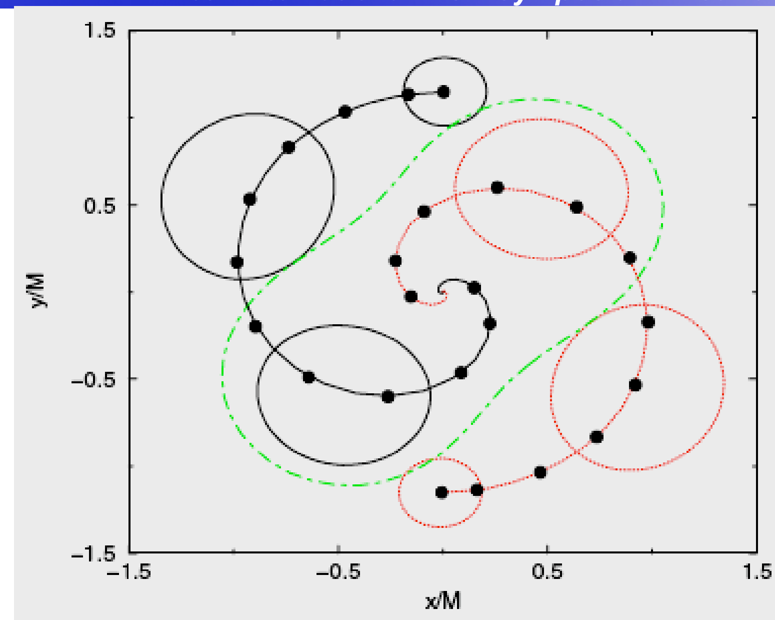
The 1st orbit, merger, & ringdown…

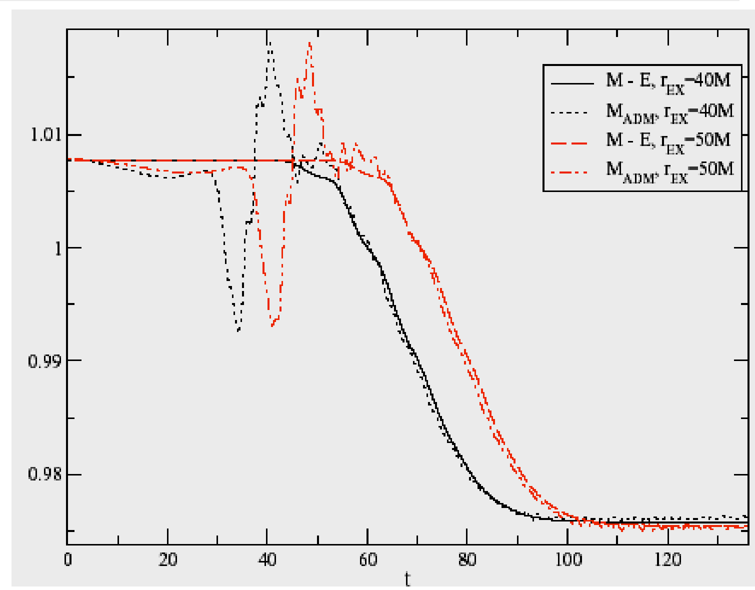
- Pretorius, PRL, 95, 121101 (2005), gr-qc/0507014
- Different formalism: based on "generalized harmonic coords"
  - metric g_{ij} is basic variable
  - 2nd order in space & time
- Excised BHs move through grid
- AMR: high resolution around BHs, tracks BHs as they move
- "Compactified" outer boundary: edge of grid at spatial infinity
- Start with 2 "blobs" of scalar field that collapse to BHs, then complete ~ 1 orbit
- Indiv BH mass M_0 ($M \sim 2M_0$)
- Show waveforms extracted at different radii (scaled)
- Re(_4) ~ d^2/dt^2 (h_+)



A new idea: “moving puncture BHs”

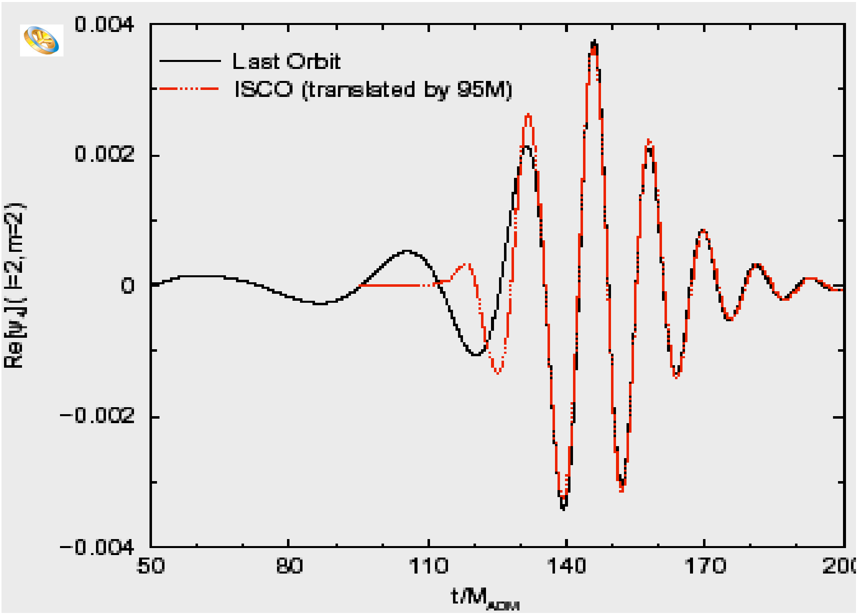
- **New techniques: move puncture BHs across grid w/out excision**
- **Simultaneous, independent discovery by UTB & GSFC groups:**
  - **Campanelli, et al., PRL, 96, 111101 (2006), gr-qc/0511048**
  - **Baker, et al., PRL, 96, 111102 (2006), gr-qc/0511103**
- **Do not split off singular part _$_{BL}$**
  - **Regularize near puncture**
  - **New conditions for _ & _^{i}**
- **Uses conformal formalism**
- **Enables long duration, accurate simulations**





A powerful new idea….

- Developed w/in the "traditional" numerical relativity approach:
  - Conformal formalism, BHs represented as punctures
- A simple, powerful new idea: allow the punctures to move
- Requires novel coordinate conditions: *Van Meter, et al., "How to move a puncture black hole without excision...," PRD, (in press, 2006), gr-qc/0605030*
- UTB, GSFC moved ahead rapidly, quickly able to do multiple orbits
- Moving punctures quickly adopted by other groups:
  - PSU, AEI/LSU, FAU/Jena…
  - At April 2006 APS meeting, a full session was devoted to BBH mergers using moving punctures!

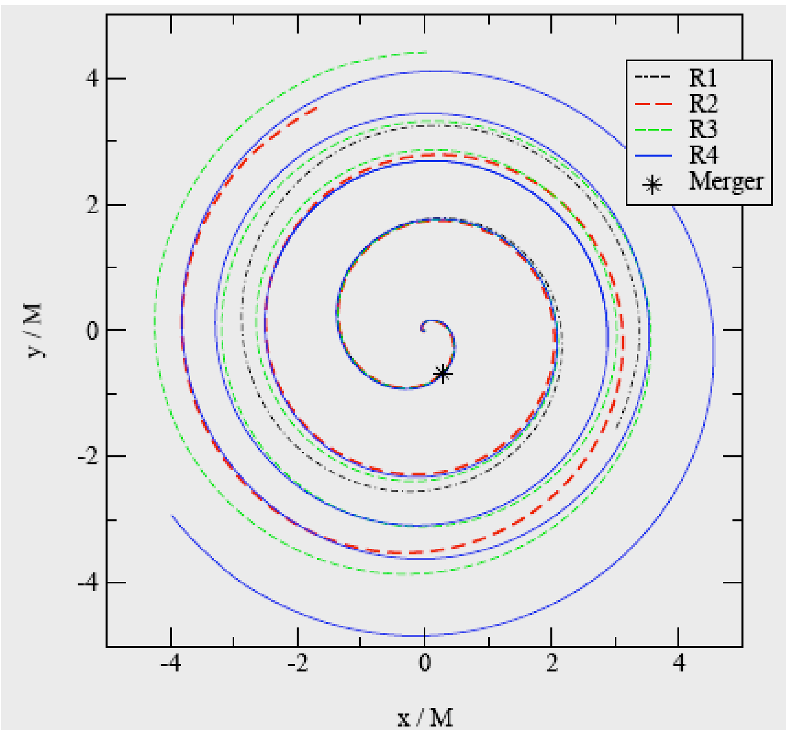


Campanelli, et al., PRD, 73, 061501 (2006), gr-qc/06010901

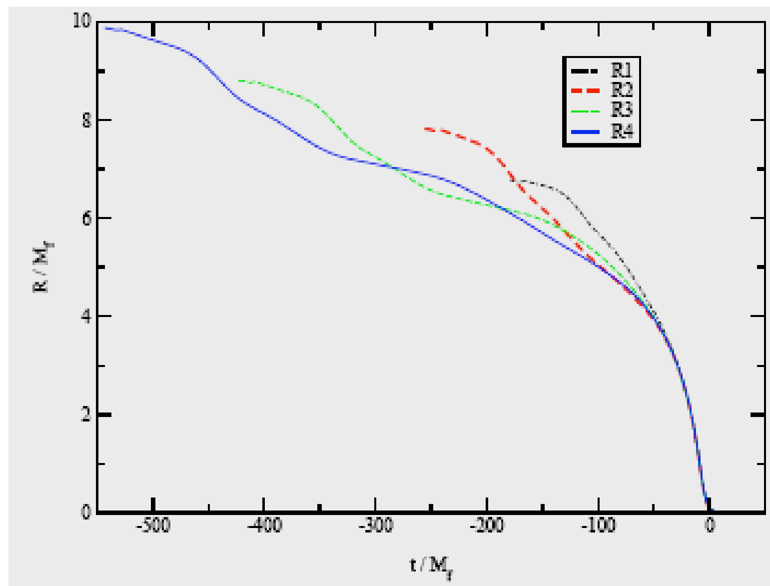
Revealing universal behavior…

- Baker, al., PRD, 73, 104002 (2006), gr-qc/0602026
- Long duration simulations of moving punctures with AMR
- Run several cases, starting from successively wider separations
- BH orbits lock on to universal trajectory ~ one orbit before merger

BH trajectories (only 1 BH shown)

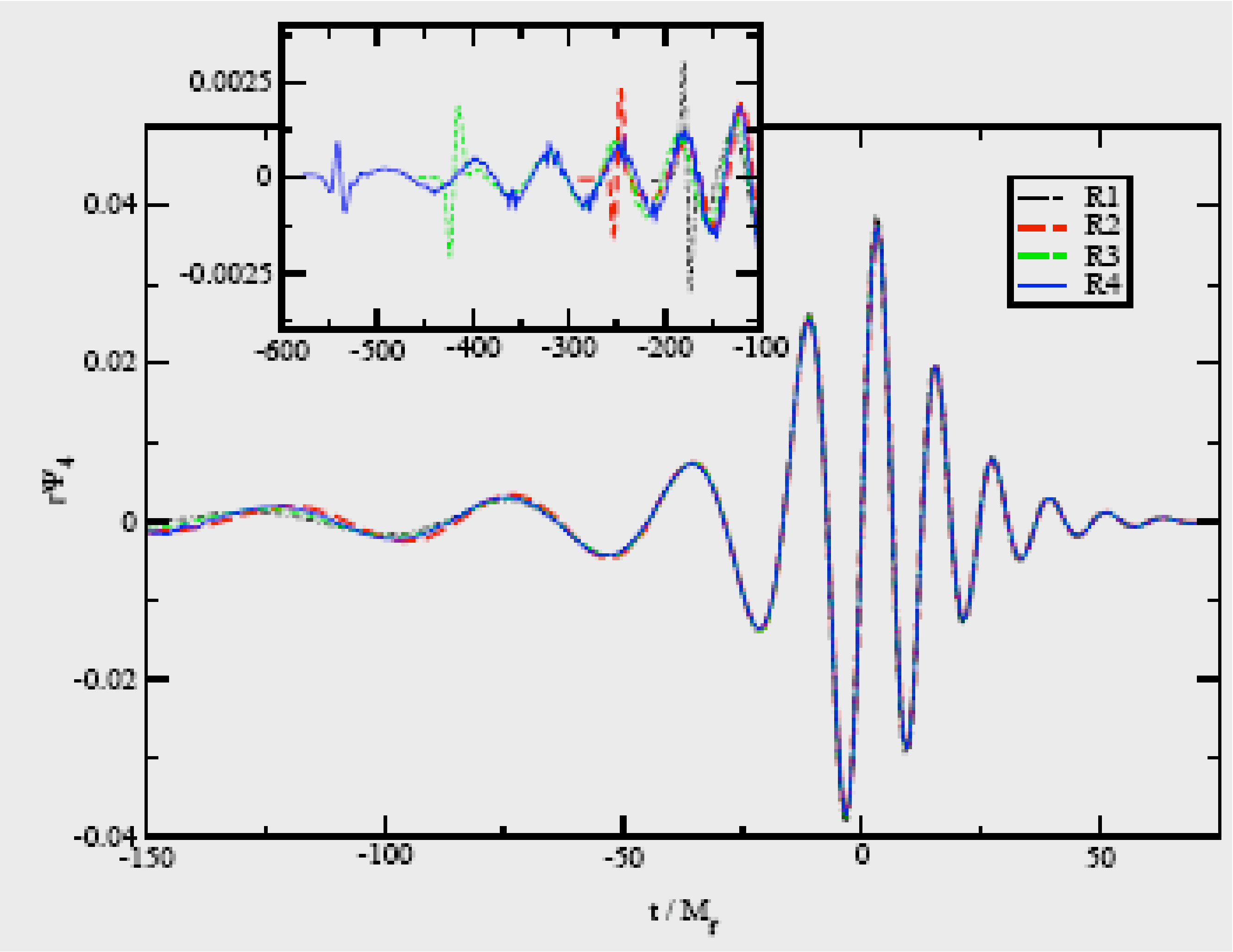


BH separation vs. time

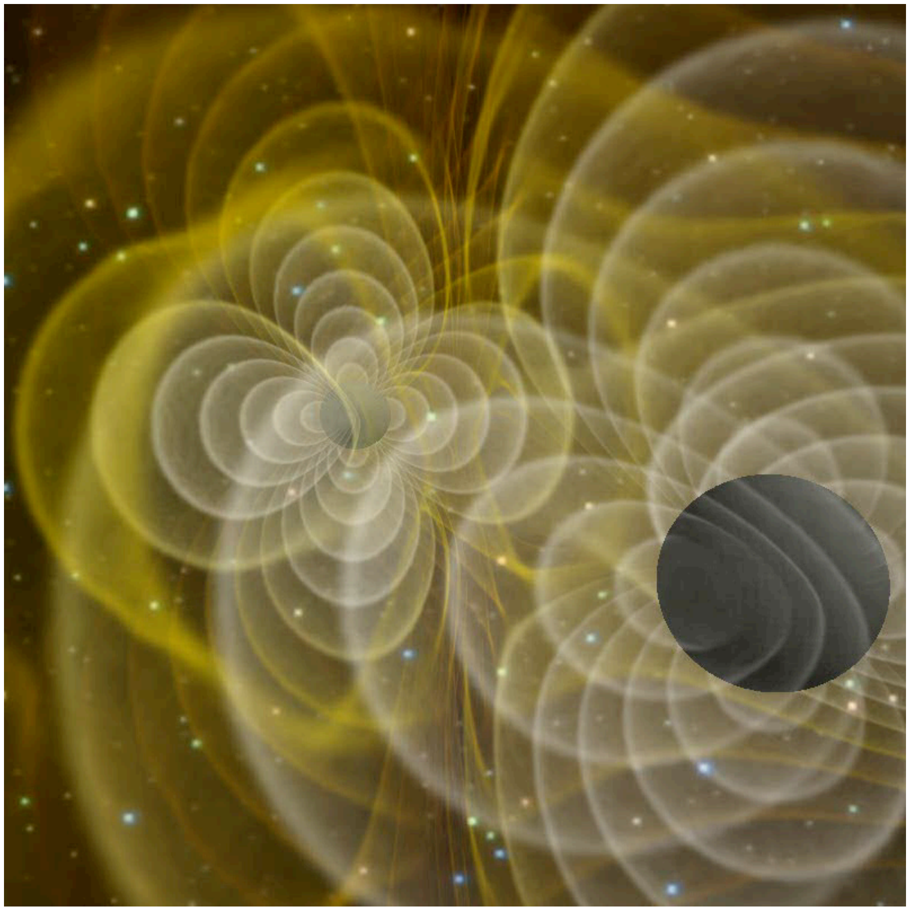


Universal waveform….

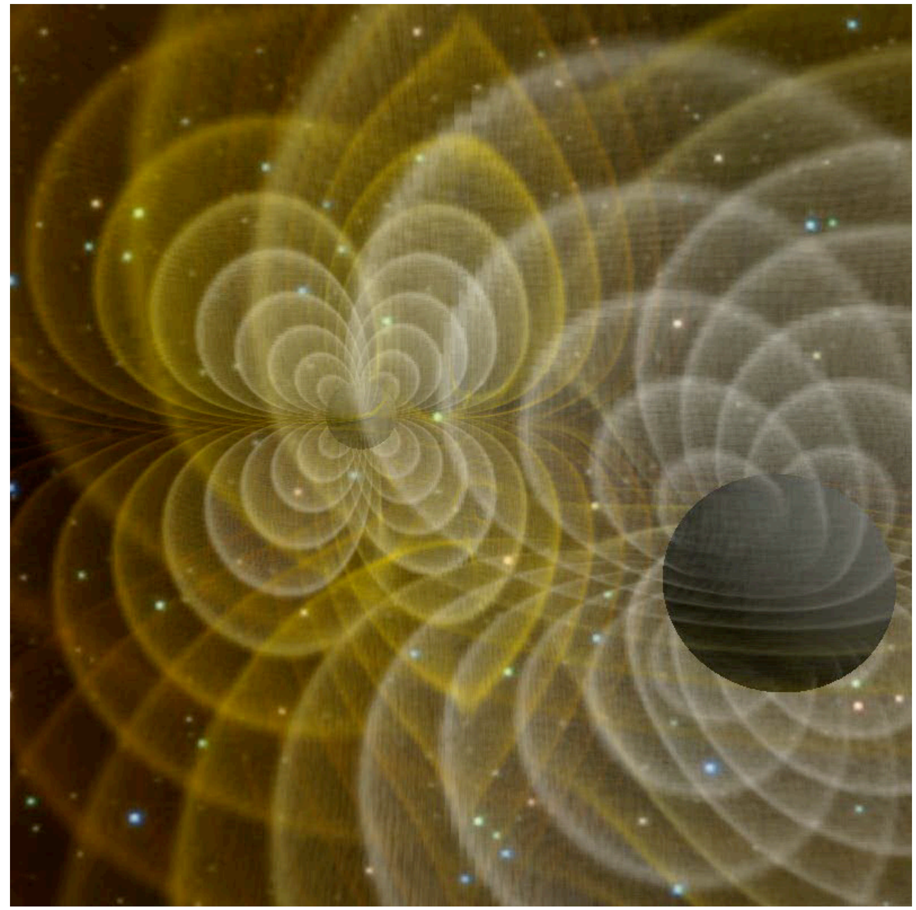
- Universal dynamics produces universal waveform....
- All runs agree to within < 1% for final orbit, merger & ringdown



BBHs: The Movies

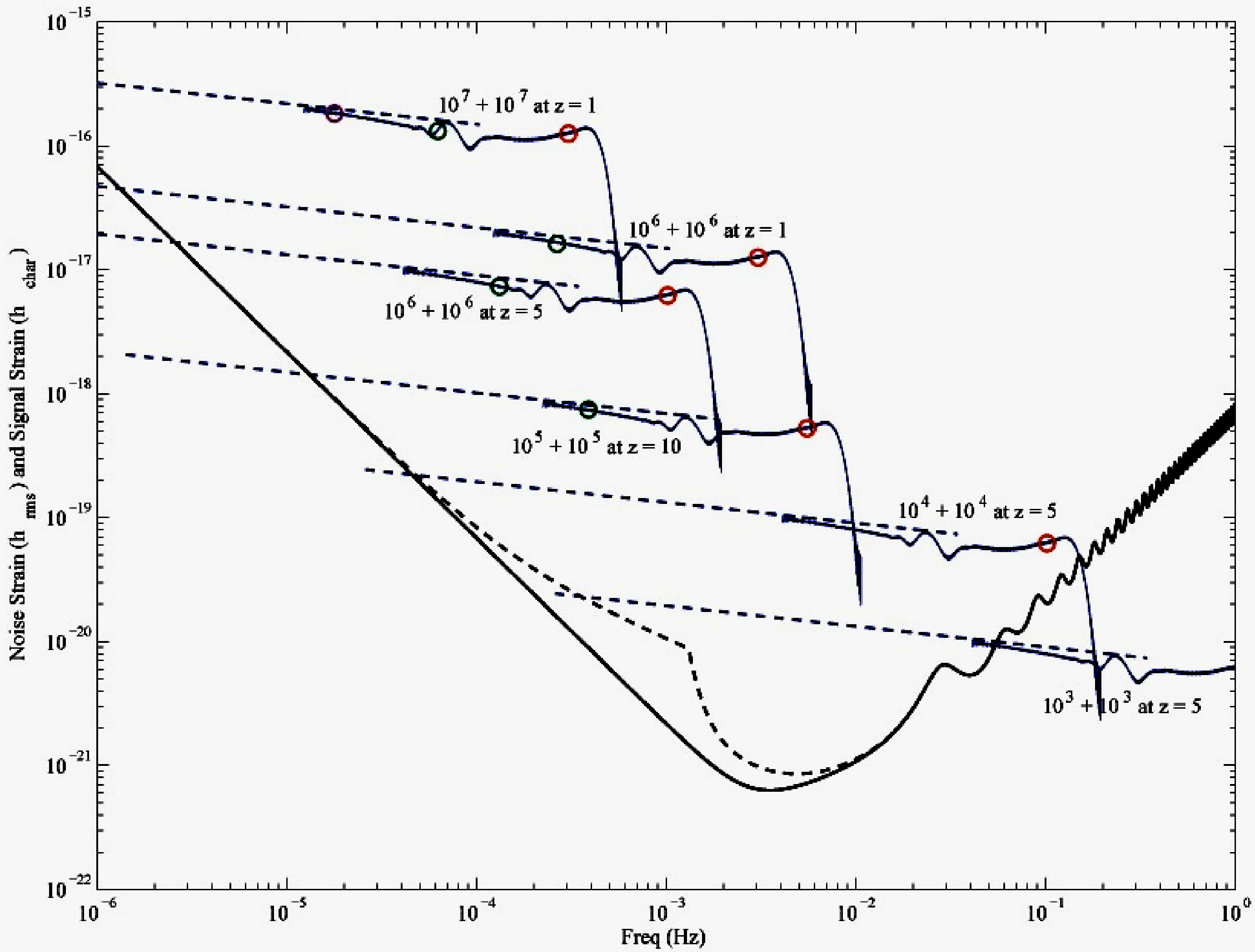


$\mathrm{Re}[\ \_{4}\ ] \sim d^2/dt^2\ h_+$



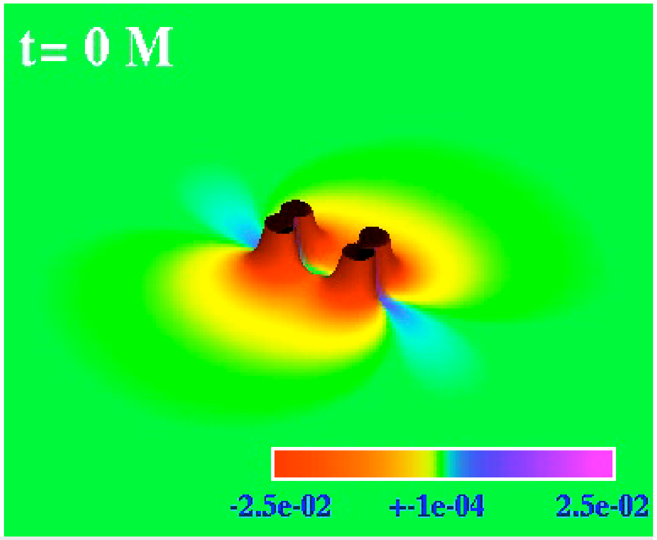
$\mathrm{Re}[\ \_{4}\ ] \sim d^2/dt^2\ h_x$

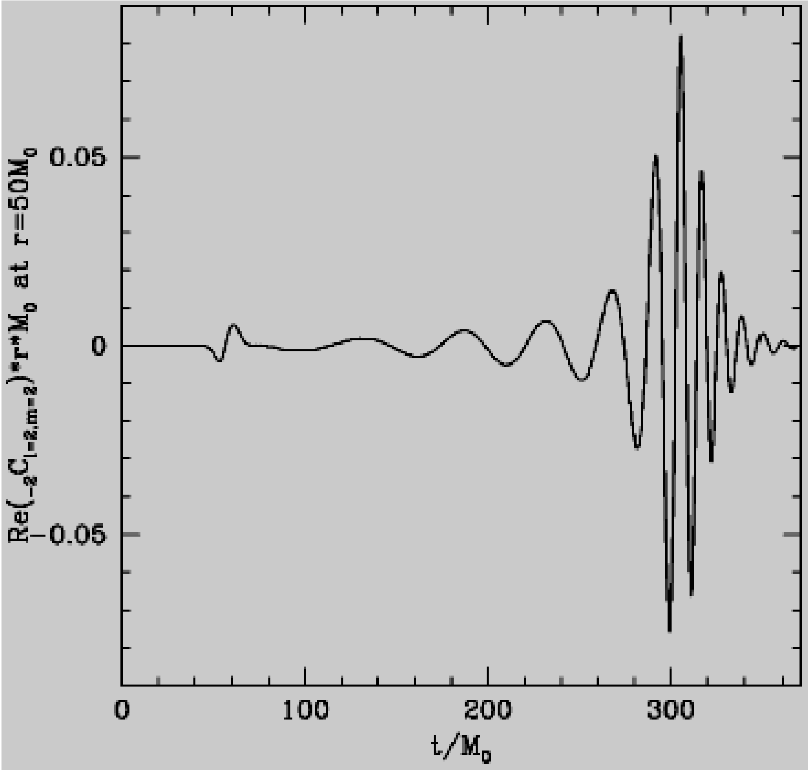
Observing with LISA….

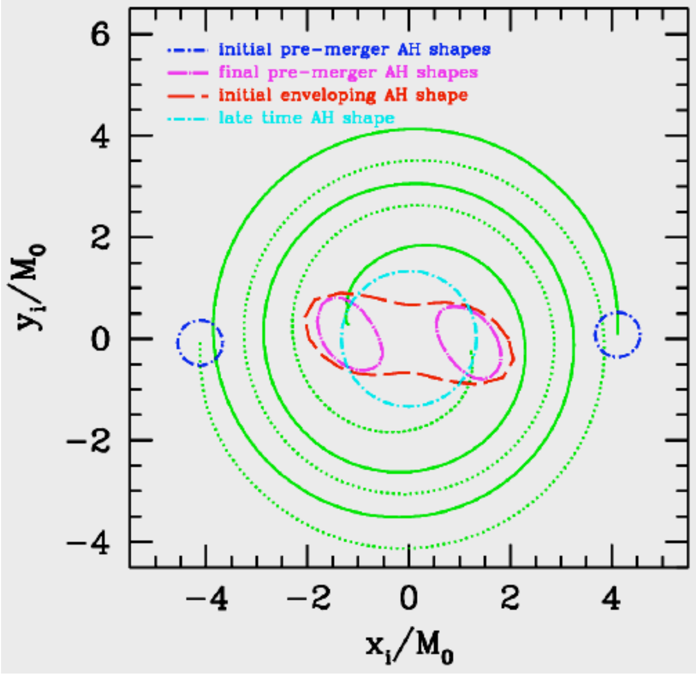


Survey of current efforts…

- Pretorius: new runs starting with more widely separated BHs
- GWs similar to those computed by other groups with moving punctures….

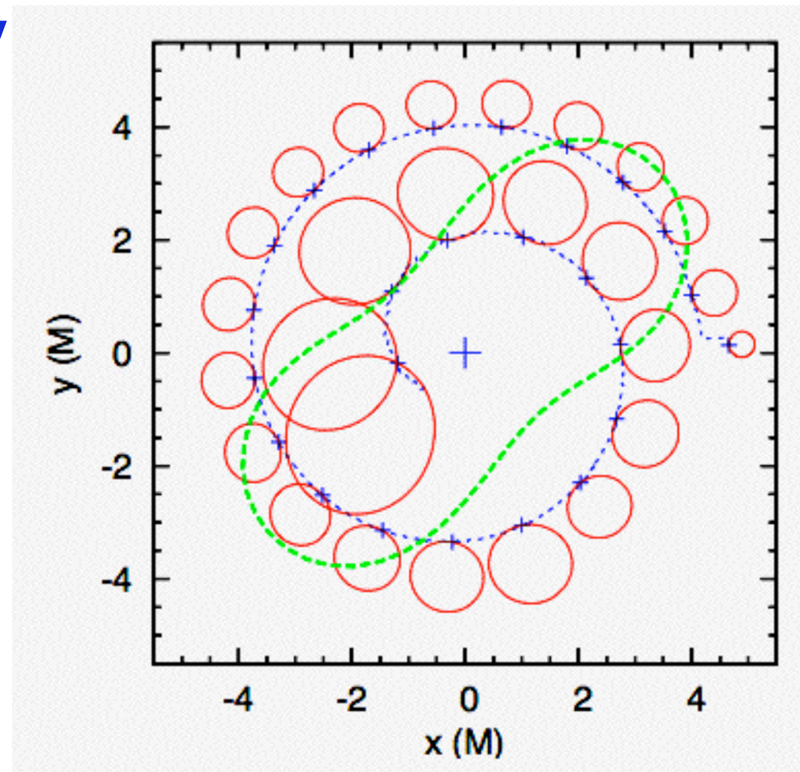






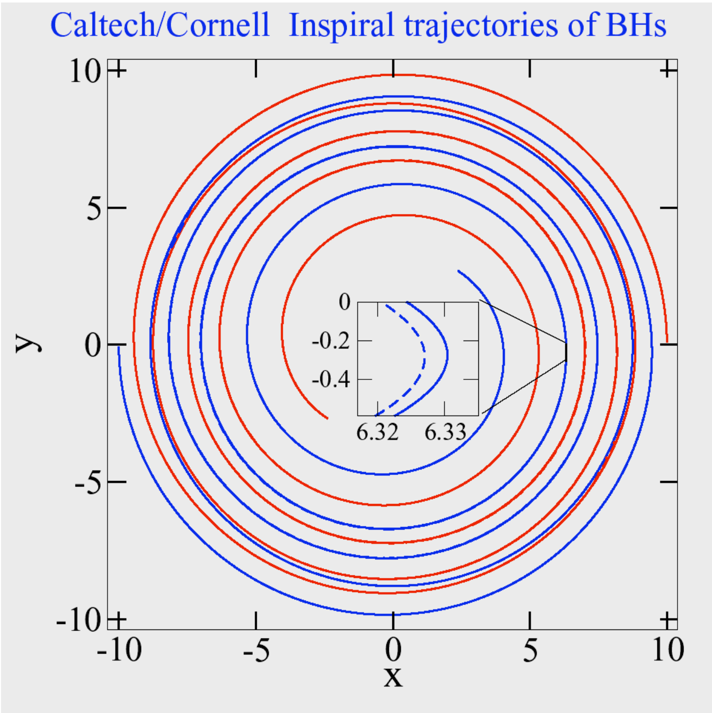
Survey of current efforts…

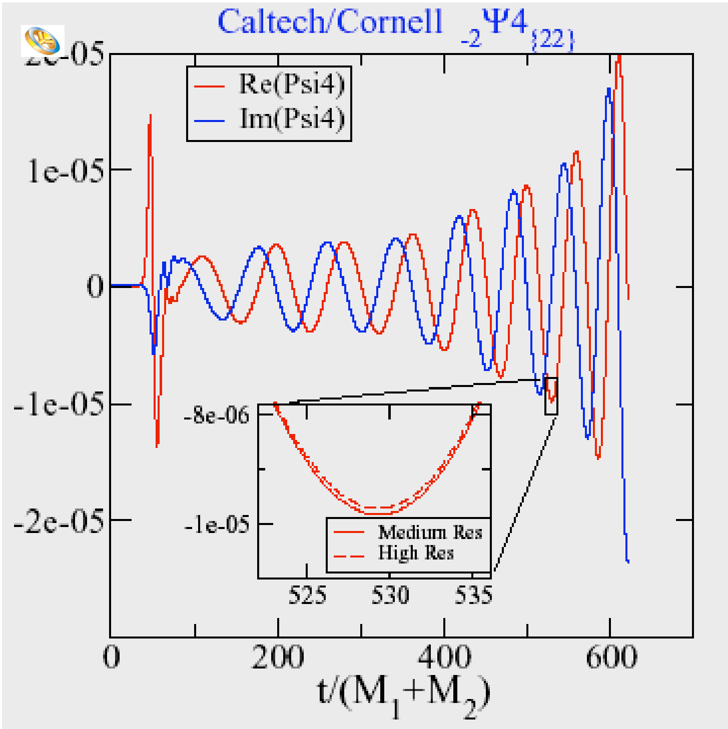
- AEI/LSU Collaboration
- Diener, et al., PRL, 96, 121101 (2006), gr-qc/0512108
- Evolve BBHs for > 1 orbit, through plunge, merger, ringdown
- 2 current research programs using conformal formalism:
  - Corotating frame with dynamically adjusted gauges
  - Moving punctures w/o excision
- GWs extracted similar to UTB and GSFC results
- Also developing code based on harmonic formulation
- Will compare results with those obtained using conformal formalism



Survey of current efforts…

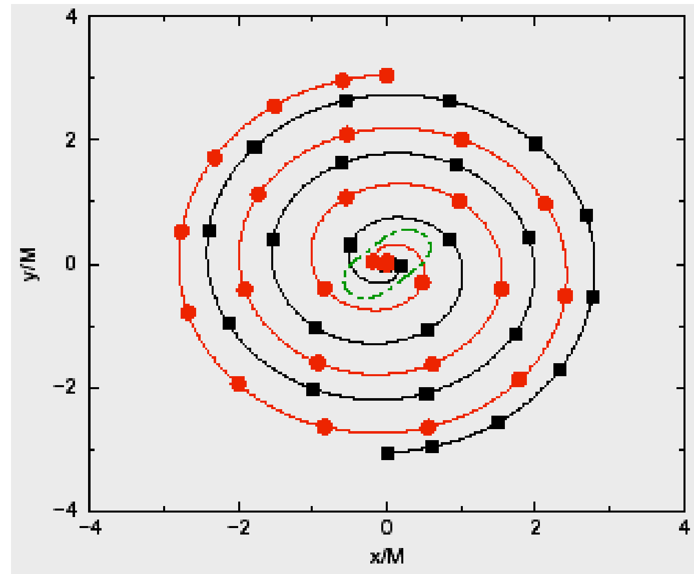
- Caltech/Cornell collaboration
- Use 1st order form of generalized harmonic formalism
- Multi-domain spectral code – very rapid convergence
- BHs are excised
- Rotating coordinates
- Currently evolve multiple orbits
- Need to re-grid to handle merger and ringdown

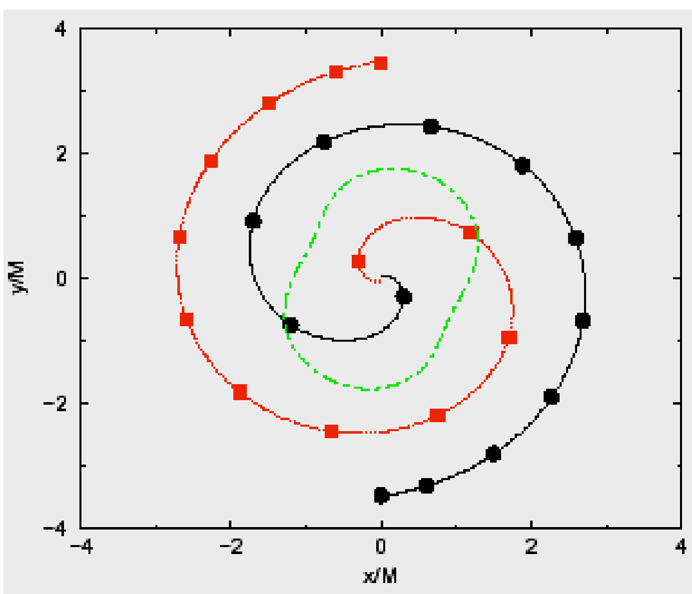


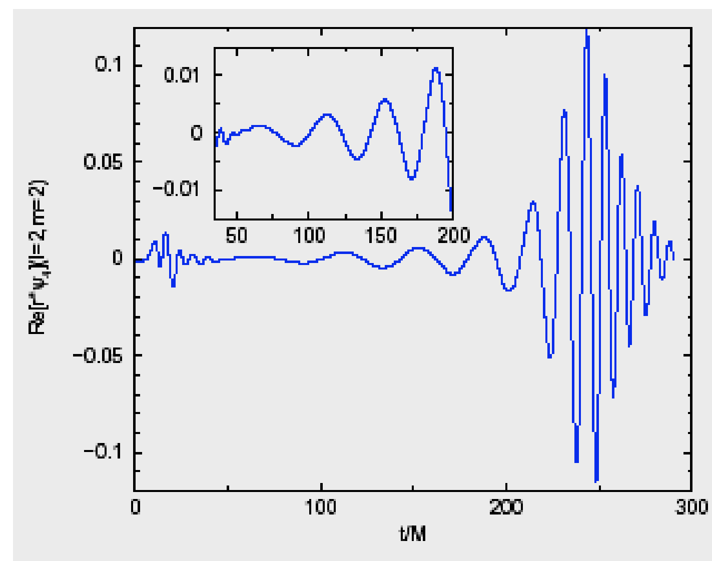


Equal mass BHs with spin…

- Campanelli, et al., gr-qc/0604012
- Moving punctures; 1^{st} BBHs with spin
- Equal masses, each with a = 0.75 m
- Initially M_ = 0.05 ◊ $T_{orbital}$ ~ 125M
- Aligned spins ◊ 'orbital hangup'
- Final a=0.9M (aligned), a=0.44M (anti)







Unequal mass BBH mergers...

- When $m_1 \neq m_2$, the GW emission is asymmetric
- GWs carry momentum, so merged remnant BH suffers a recoil ‘kick’
- Most of the recoil occurs in strong gravity regime ◊ requires numerical relativity simulations
- Unequal mass mergers are technically more demanding
- Herrmann, et al., gr-qc/0601026: 1st unequal mass BBH simulations, use moving puncture method
  - gives lower limits on kicks
- Baker, et al., astro-ph/0603204: used wider separations, higher resolution, AMR

Herrman, et al.	
m_1/m_2	V (km/s)
1.00	1 ± 1
0.85	49 ± 11
0.78	69 ± 19
0.55	82 ± 27
0.32	25

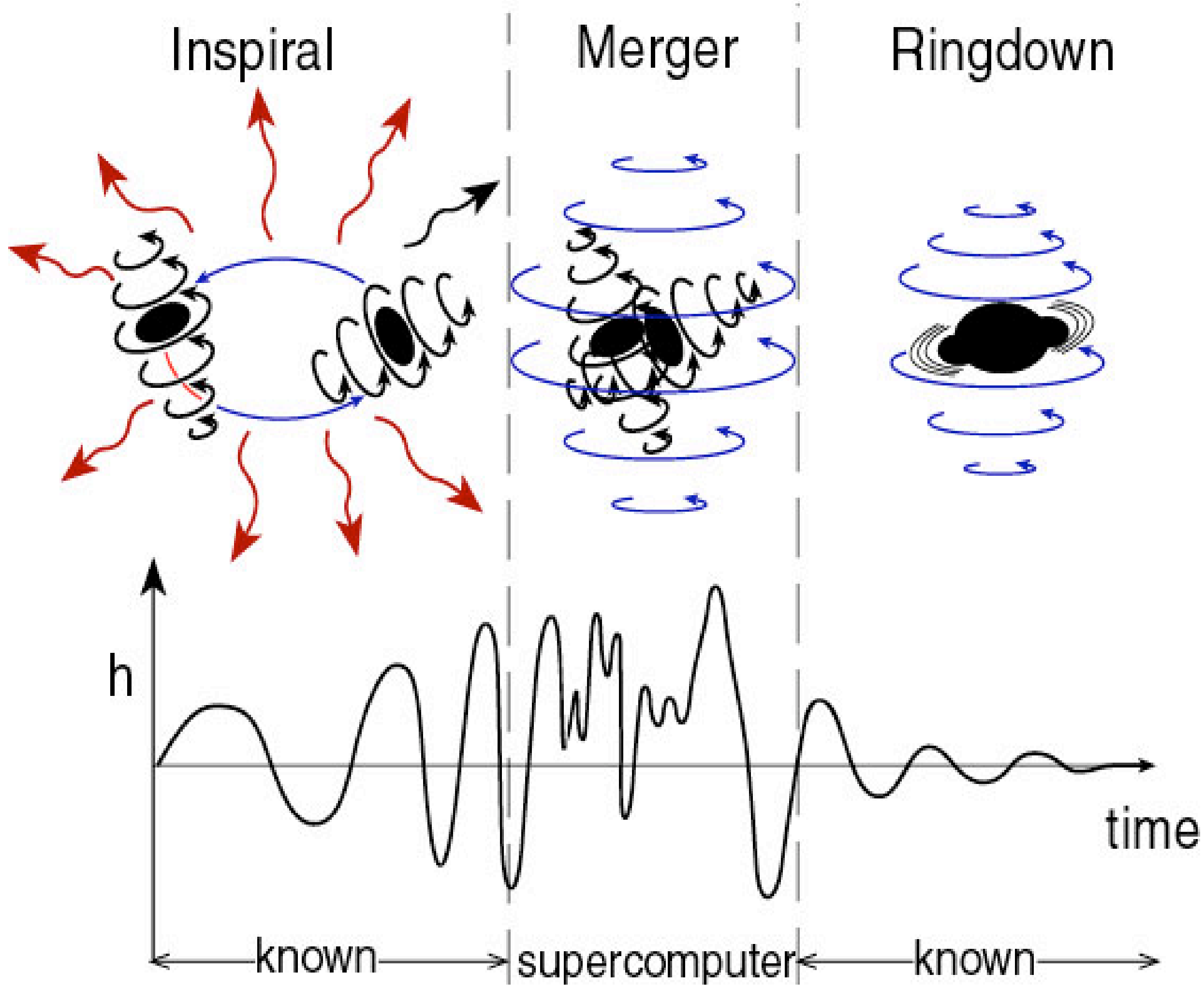
Baker, et al.	
m_1/m_2	V (km/s)
0.67	105 ± 10

Status of BBH merger simulations

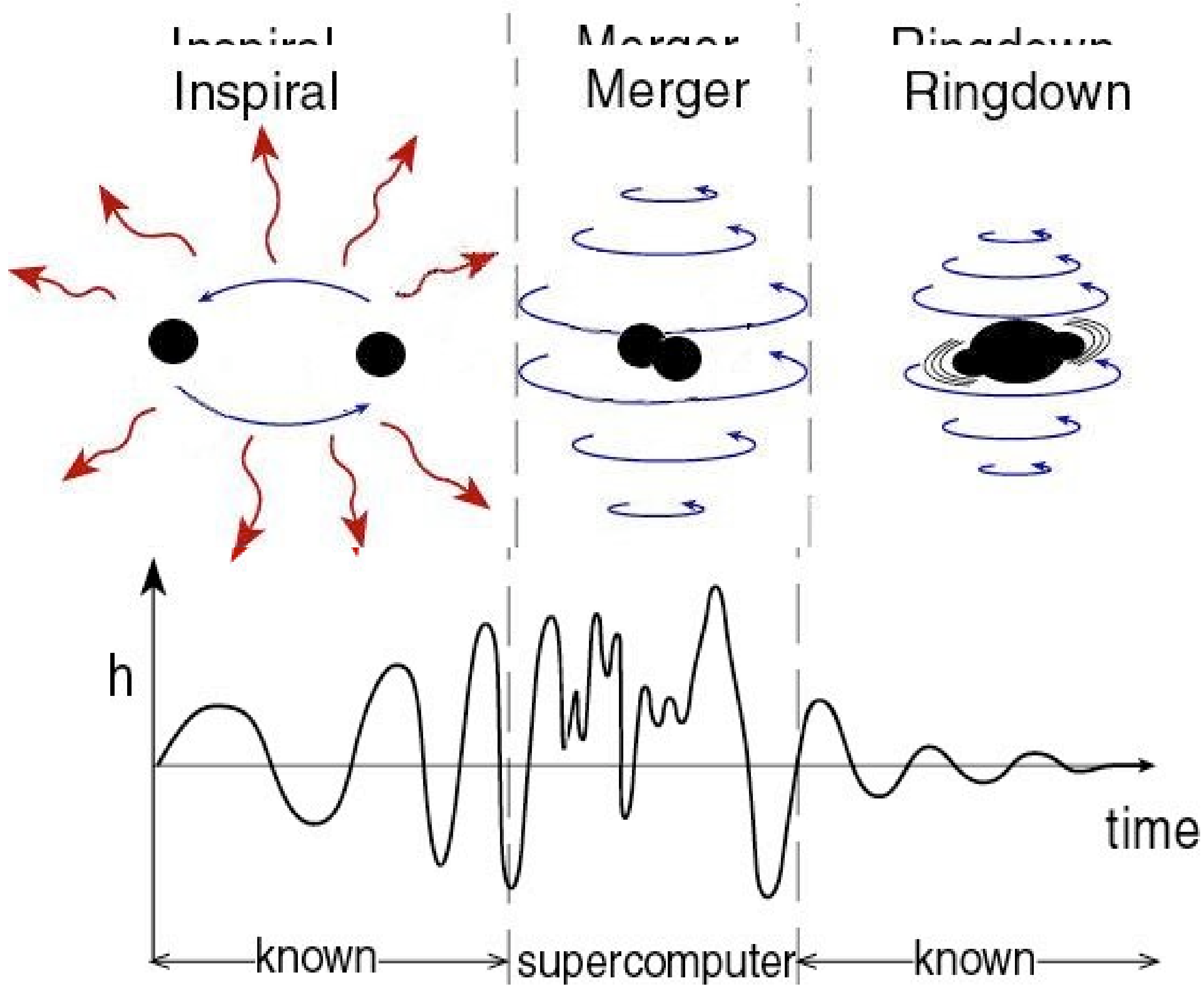
- Impressive progress on a broad front: many research groups, different codes, methods…
- Equal mass, nonspinning BBHs: several groups are now capable of evolving for several orbits, followed by the plunge, merger, and ringdown
- There is general agreement on the simple waveform shape and that
  - Total GW energy emitted in last few cycles _E ~ (0.035 – 0.04)M *(depending on how many orbits are in the simulation)*
  - Final BH has spin a ~ 0.7M
- Efforts currently underway to compare waveform results from simulations by UTB, GSFC, and Pretorius
- This will expand to include other groups in the community…
- Work has begun on BBHs with unequal masses, and with spins

The emerging picture….

The emerging picture….



The emerging picture….




Stay Tuned!

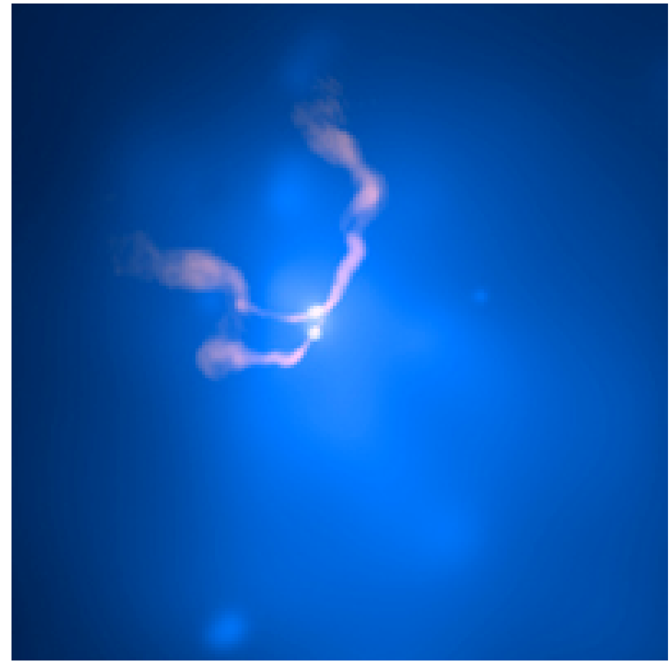
Observing the final moments of massive black hole mergers with LISA

John Baker

Gravitational Astrophysics Laboratory
NASA/GSFC

Black Hole Mergers

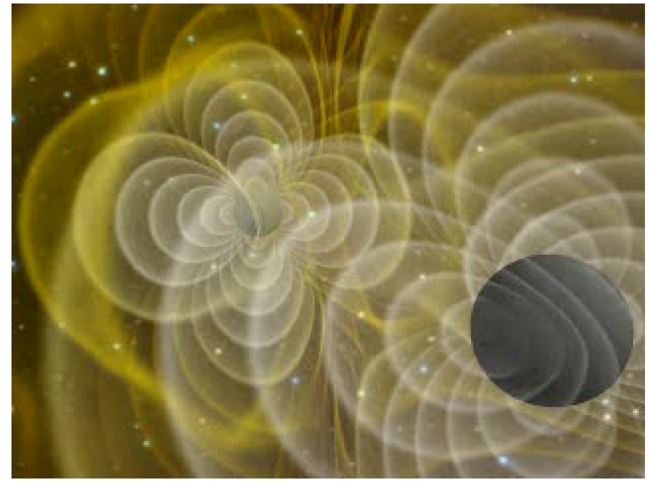
- LISA: sensitive to MBH mergers
  - MBHs found in galactic centers
  - MBH mergers trace galactic mergers
  - Likely ~ few per year
  - Merger observations may provide:
    - Knowledge of structure formation
    - Tests of GR
  - Last few cycles ("merger")
    - May generate most of SNR
    - Requires numerical simulation
- Ground-based
  - Sensitive to stellar-scale BH mergers



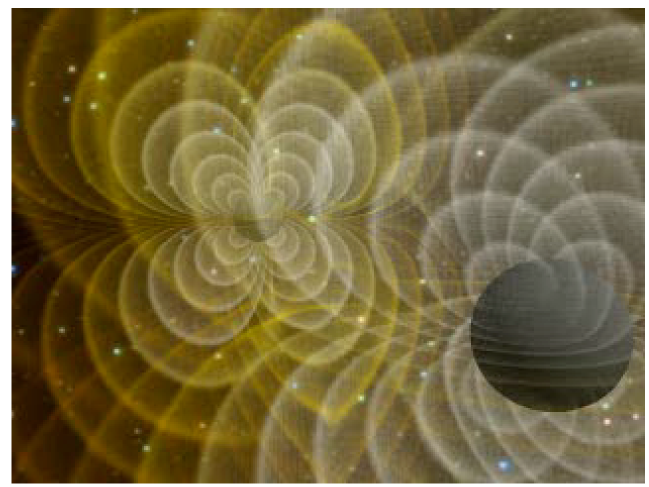
Composite X-ray/radio image of 3C 75 in Abell 400 cluster.
Credit: X-ray: NASA/CXC/AIfA/D.Hudson & T.Reiprich et al.;
Radio: NRAO/VLA/NRL

Black hole merger modeling

- Simulation of vacuum space-time dynamics
  - Set initial data with BHs
  - Numerically evolve full Einstein’s equations
  - Simulations scale with total M
- Evolution now works!
  - Recent improvements in coordinate conditions
  - Changes (simplification) in handling BHs numerically
  - Higher accuracy:
    - Higher-order finite differencing
    - AMR
    - Spectral methods
  - See other talks…(session Q11)
    - D. Choi, M. Koppitz, J. Van Meter

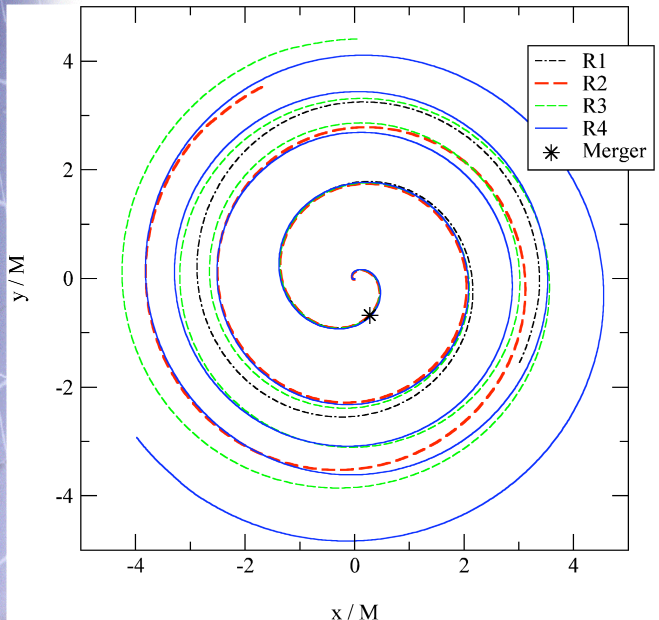


Re[_$_4$] ~ $d^2/dt^2\ h_+$

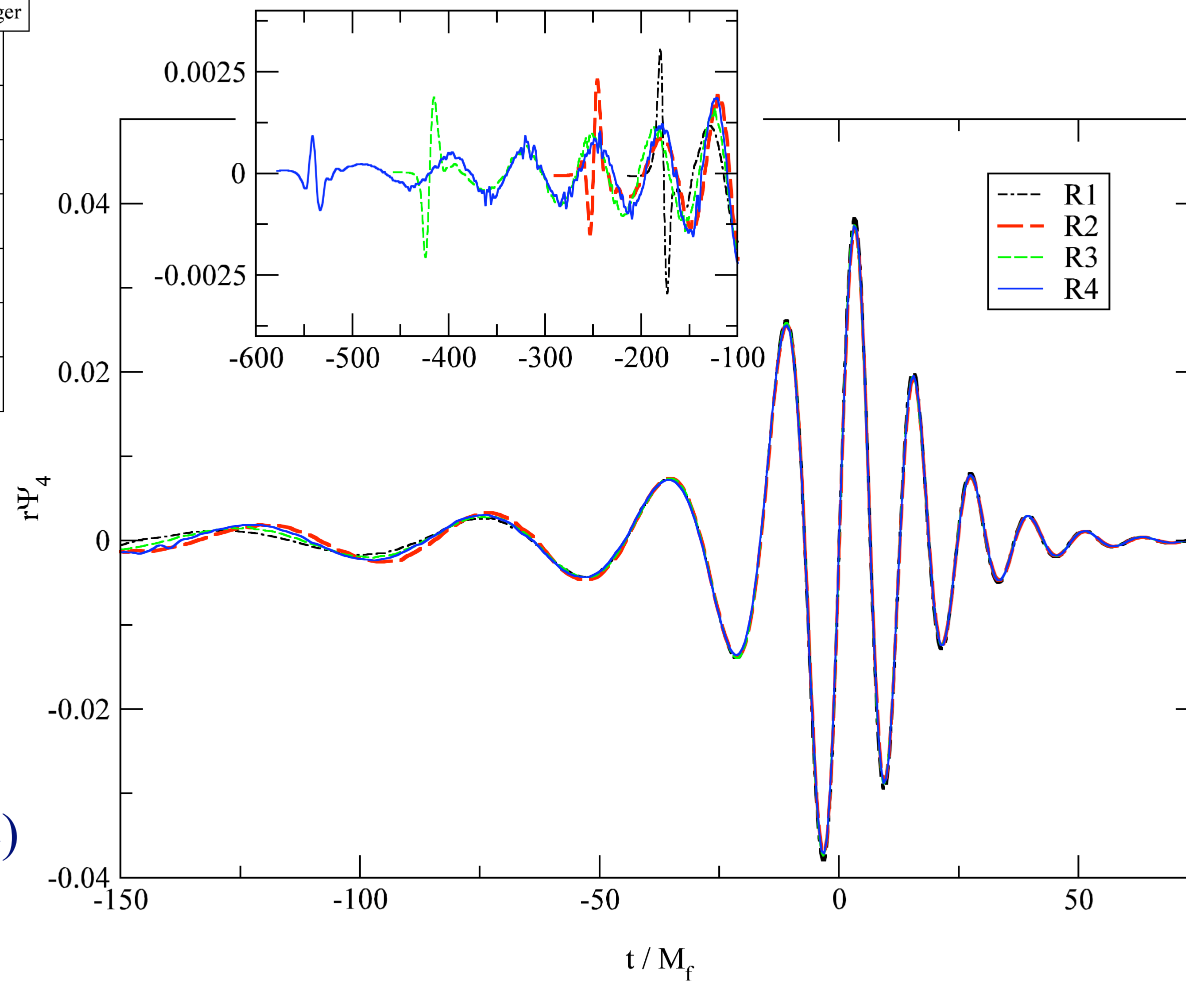


Im[_$_4$] ~ $d^2/dt^2\ h_x$

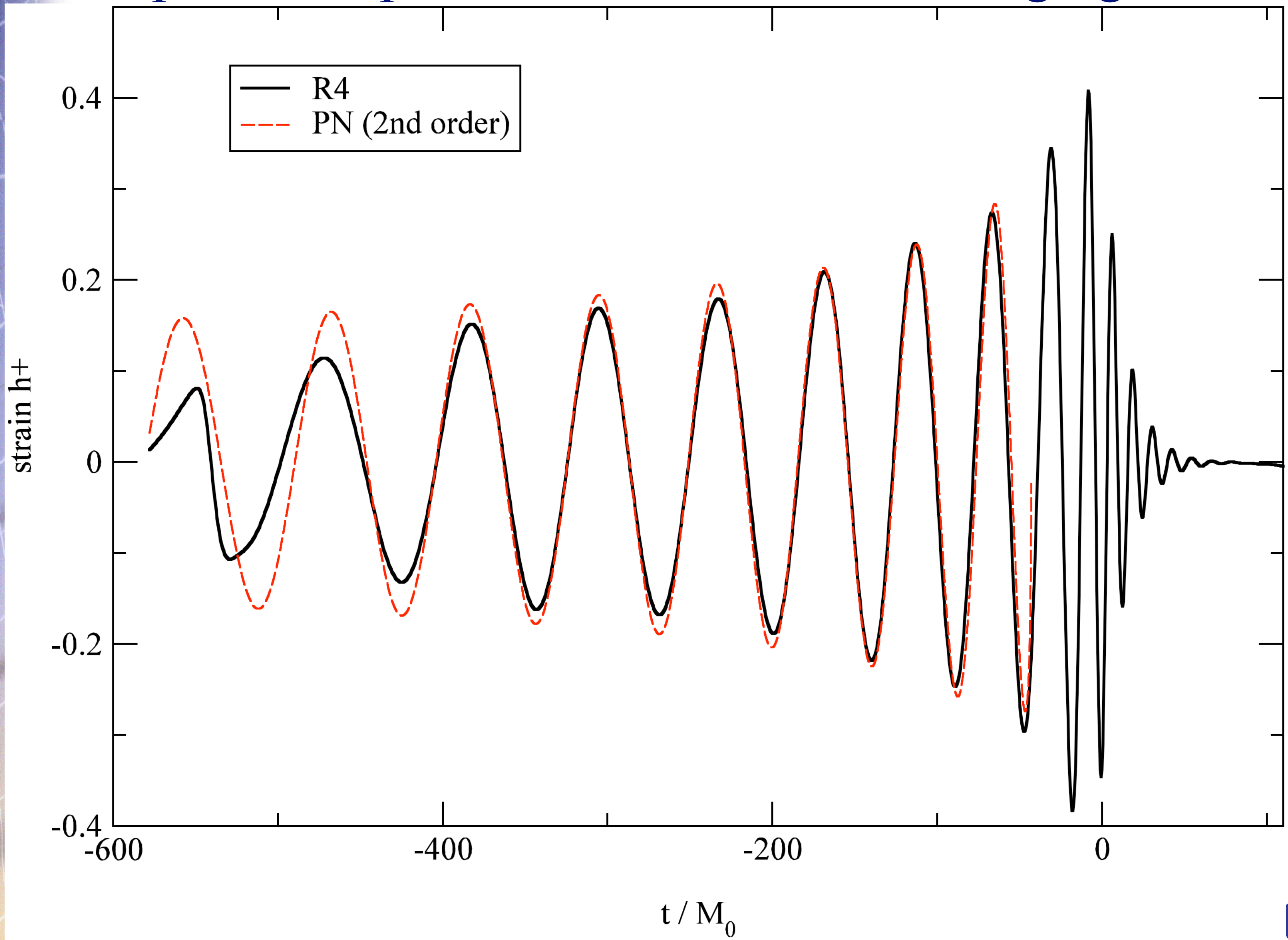
Results: Equal-mass non-spinning mergers



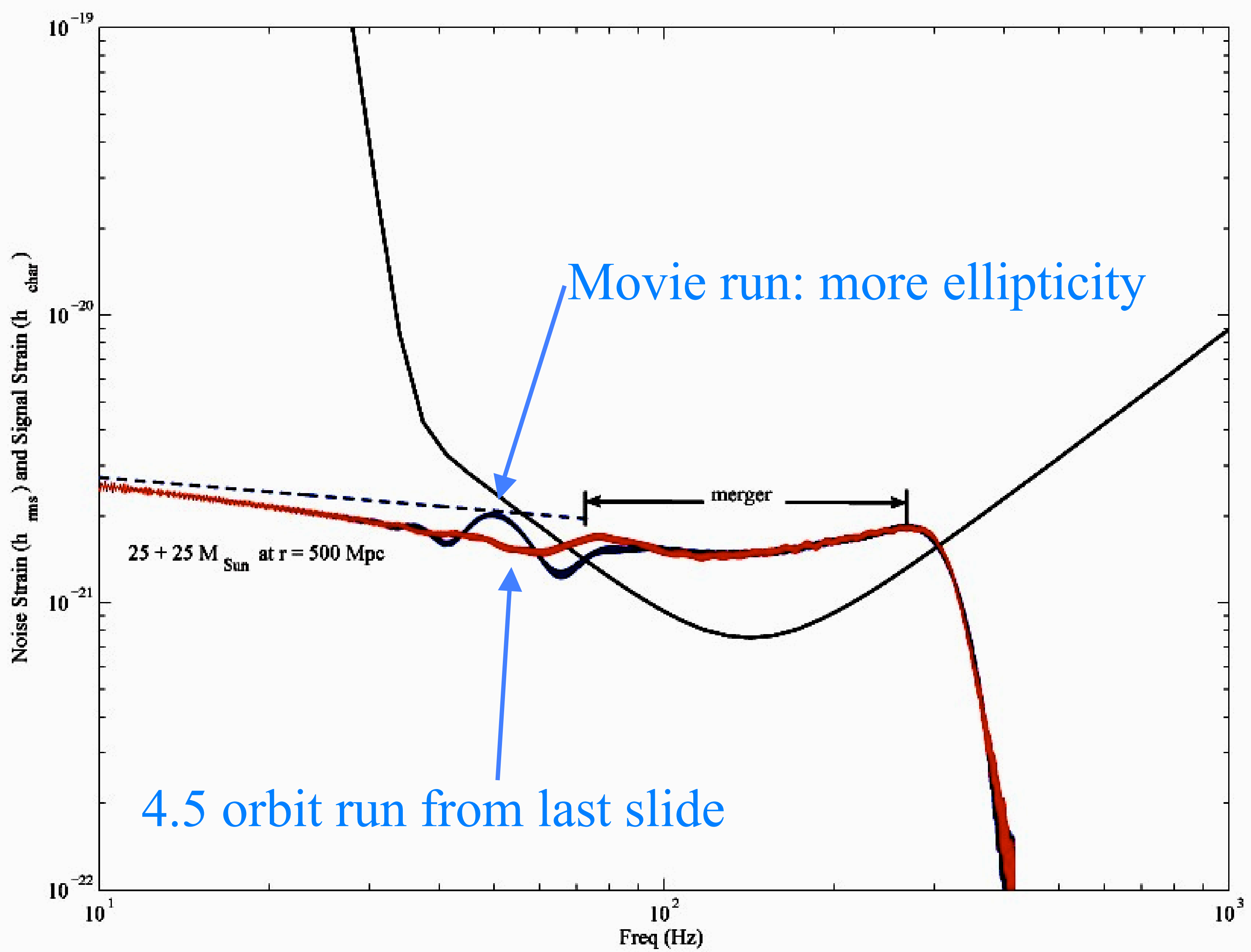
- Four simulations
  - $10\ M < L_0 < 13\ M$
  - 1.5 to 4.5 orbits
  - $\Delta E \sim 3.5$ — 4
    (3-4 times inspiral ΔE)
- Agreement to ~1% agreement for late burst !
- General agreement earlier



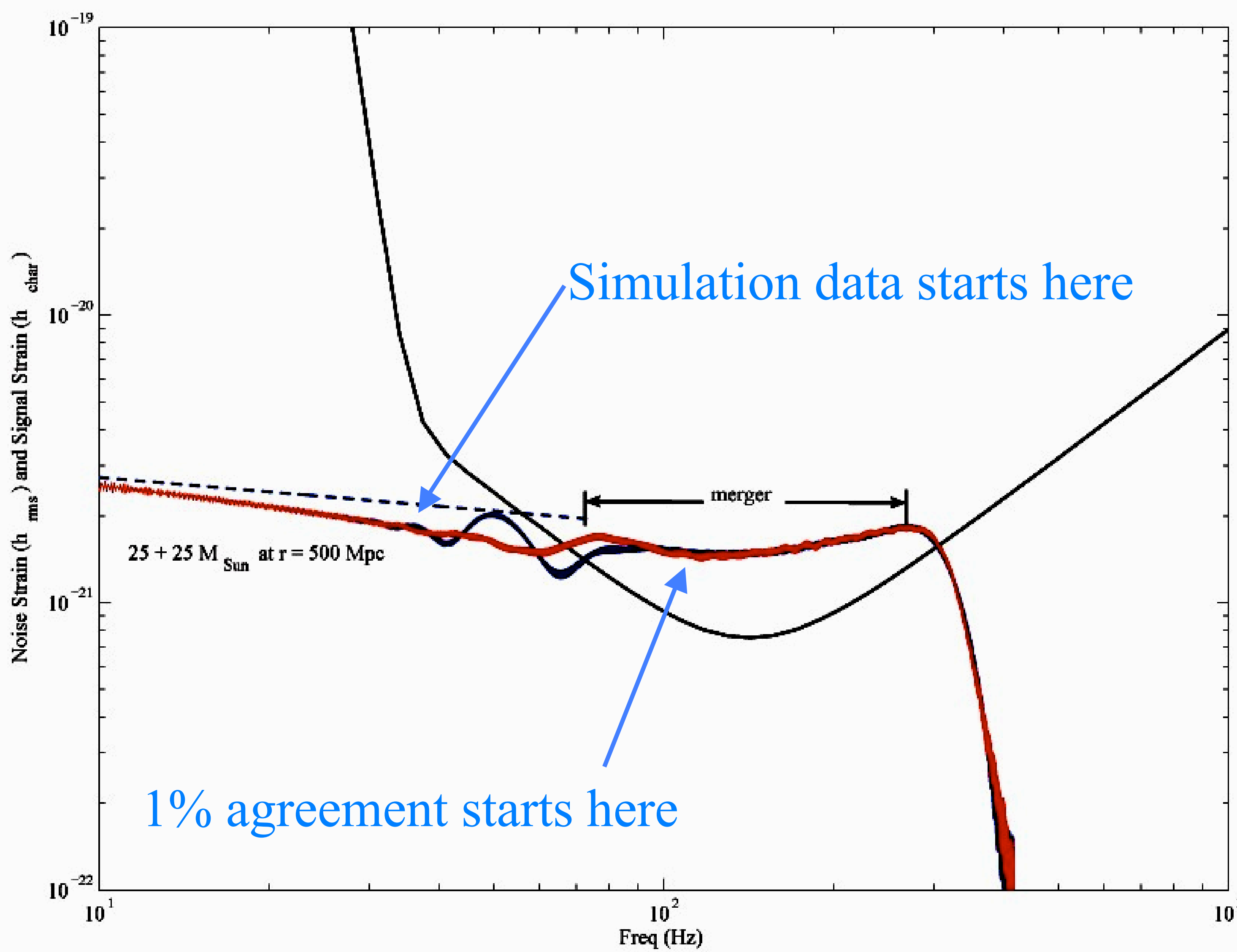
Compare with post-Newtonian… encouraging



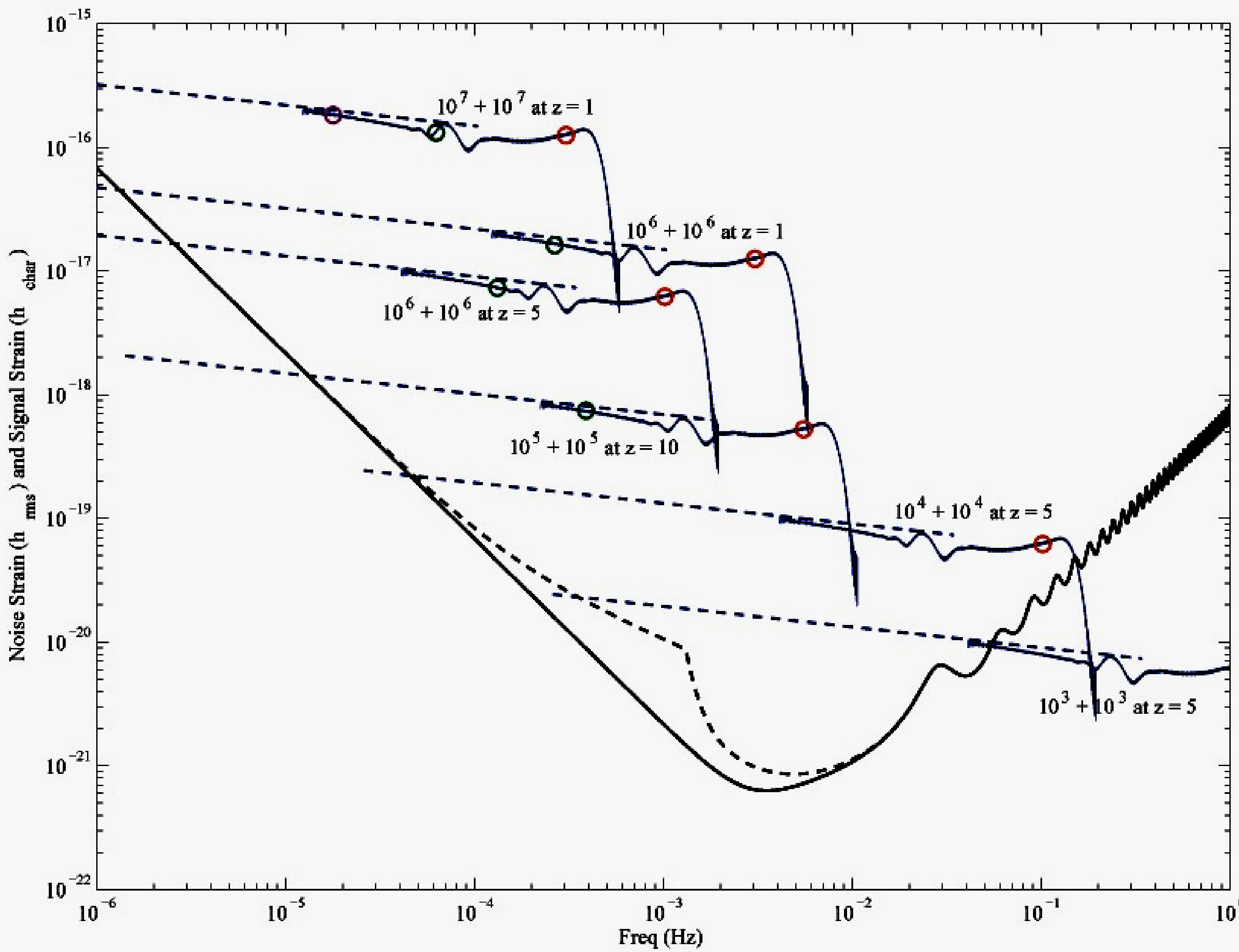
As seen by LIGO… (credit; Sean Mc Williams)



As seen by LIGO… (credit; Sean Mc Williams)

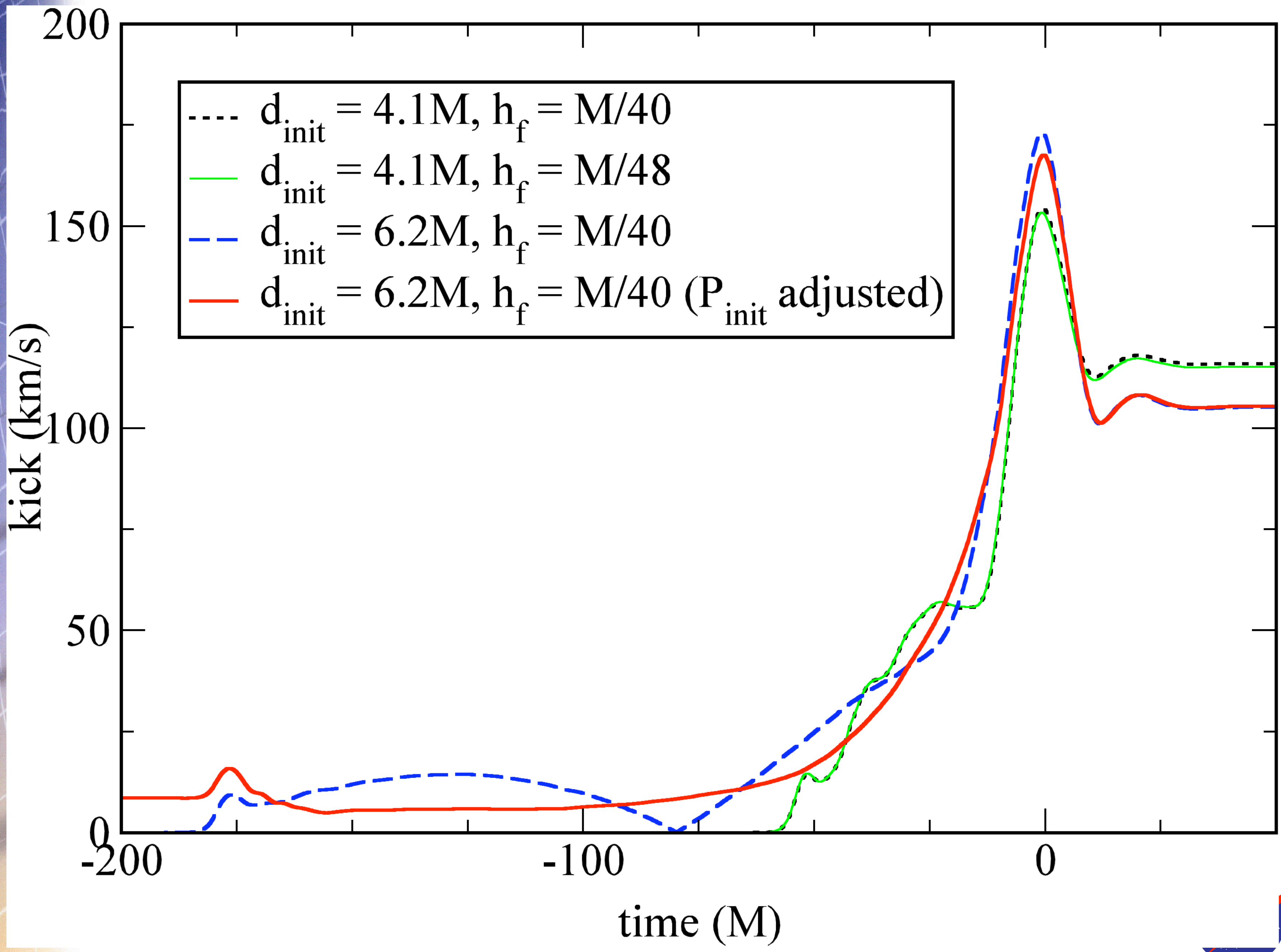


Observing with LISA (after)... (Sean McWilliams)



Ongoing and future work

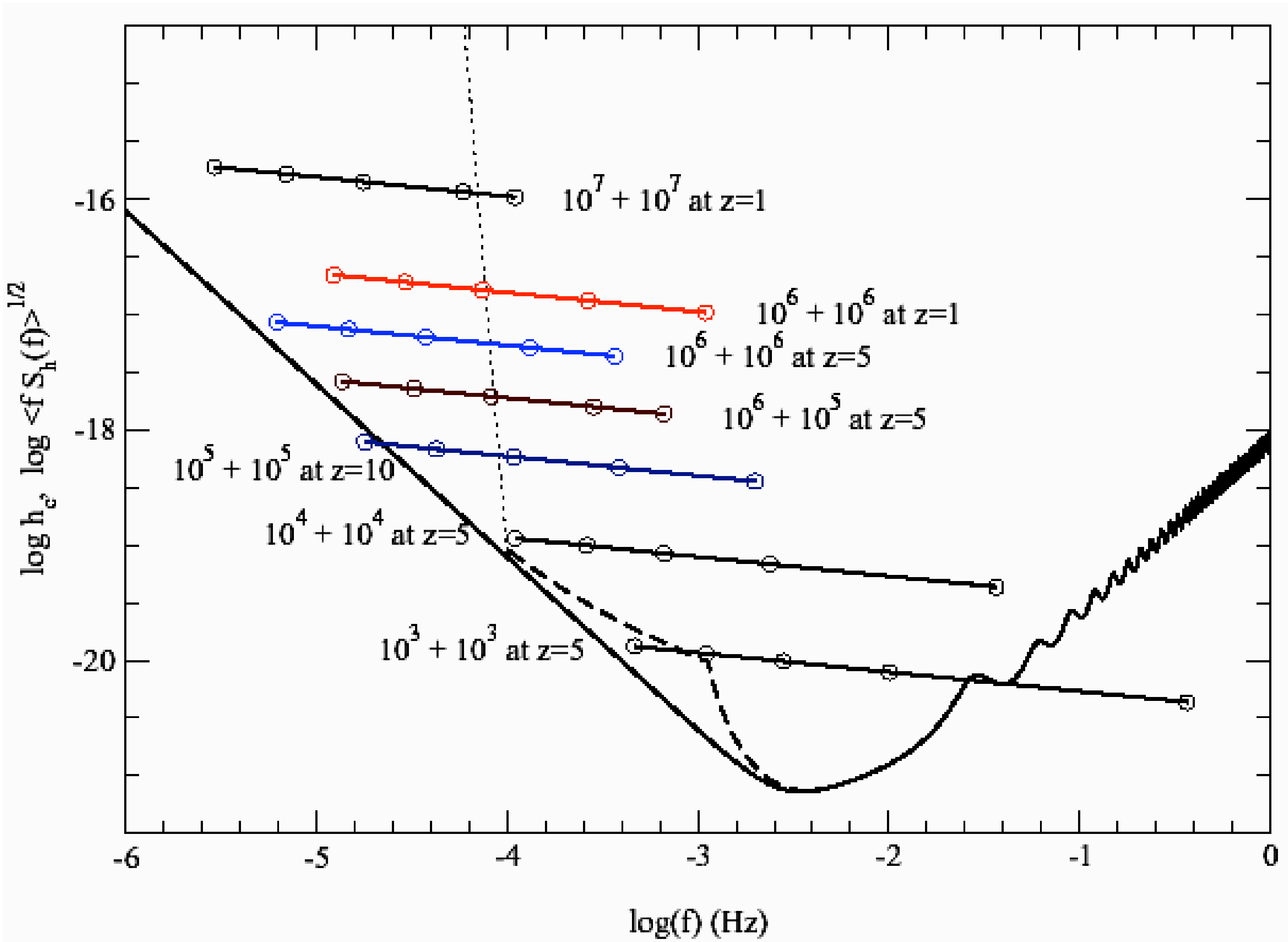
- Toward complete waveforms
  - For accuracy lasting 1000 M
    - Initial BH configuration
    - Qualify numerical techniques
- Toward enhanced parameter estimation with LISA
  - Explore parameter space
    - Mass-ratio
      - Results at $M_1/M_2=1.5$
      - Beyond $M_1/M_2=2\text{-}3$ more challenging
    - Spins
      - Beyond a/m=.7 more challenges
      - UTB results
      - Precessing systems more demanding
  - Empirical fit to cover parameter variations
- Toward BBH merger LISA science
  - How to test GR?
  - Numerical simulation input to astrophysics
    - Final spin
    - Gravitational radiation "kick" … see D. Choi



d_{init} = 4.1M, h_f = M/40
d_{init} = 4.1M, h_f = M/48
d_{init} = 6.2M, h_f = M/40
d_{init} = 6.2M, h_f = M/40 (P_{init} adjusted)
kick (km/s)
time (M)
200
150
100
50
0
-200
-100
0

MBH binary inspirals and LISA (before)

- symbols at 10 yrs, 1 yr, 1 mo, & 1 d before the onset of merger, and at the onset of merger (the merger & subsequent ringdown occurs at higher frequencies)



Simulation of Binary Black Hole Mergers

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University of Alberta

CITA

April 20, 2006

Outline

- Why study binary black hole systems?
  - expected to be among the strongest and most promising sources of gravitational waves that could be observed by gravitational wave detectors
  - understand the strong-field regime of general relativity
- Why do we need to *simulate* them?
  - understanding the nature of the gravitational waves emitted during a merger event may be *essential* for successful detection
  - the two-body problem in GR is *unsolved*, and no analytic solution techniques (perturbative or other) known that could be applied during the final stages of an inspiral and merger
- Methodology
  - brief overview of numerical relativity, the difficulties in discretizing the field equations
- Simulation results
  - evolution of quasi-circular initial data sets
  - binaries constructed via scalar field collapse

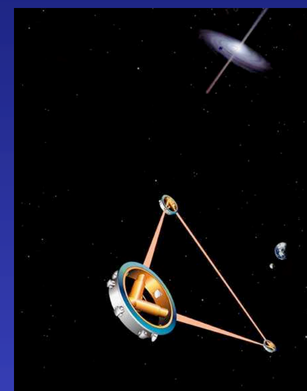
The network of gravitational wave detectors

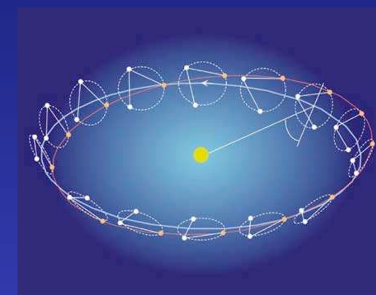
LIGO/VIRGO/GEO/TAMA
ground based laser interferometers

LIGO Livingston

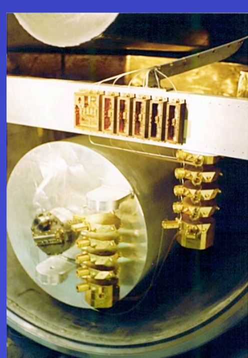
LIGO Hanford

LISA
space-based laser interferometer (hopefully with get funded for a 201? Lauch)





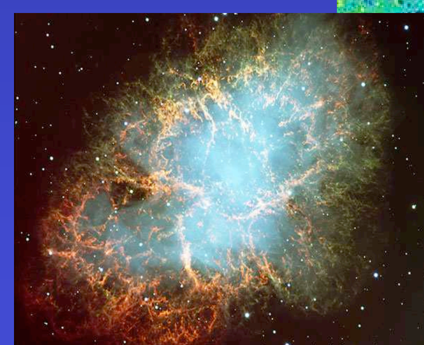
ALLEGRO/NAUTILUS/AURIGA/…
resonant bar detectors



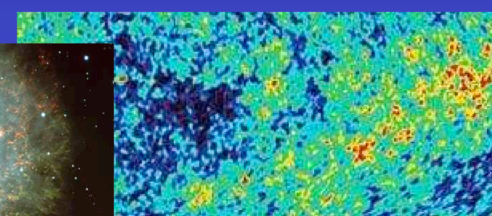
ALLEGRO

AURIGA

Pulsar timing network, CMB anisotropy

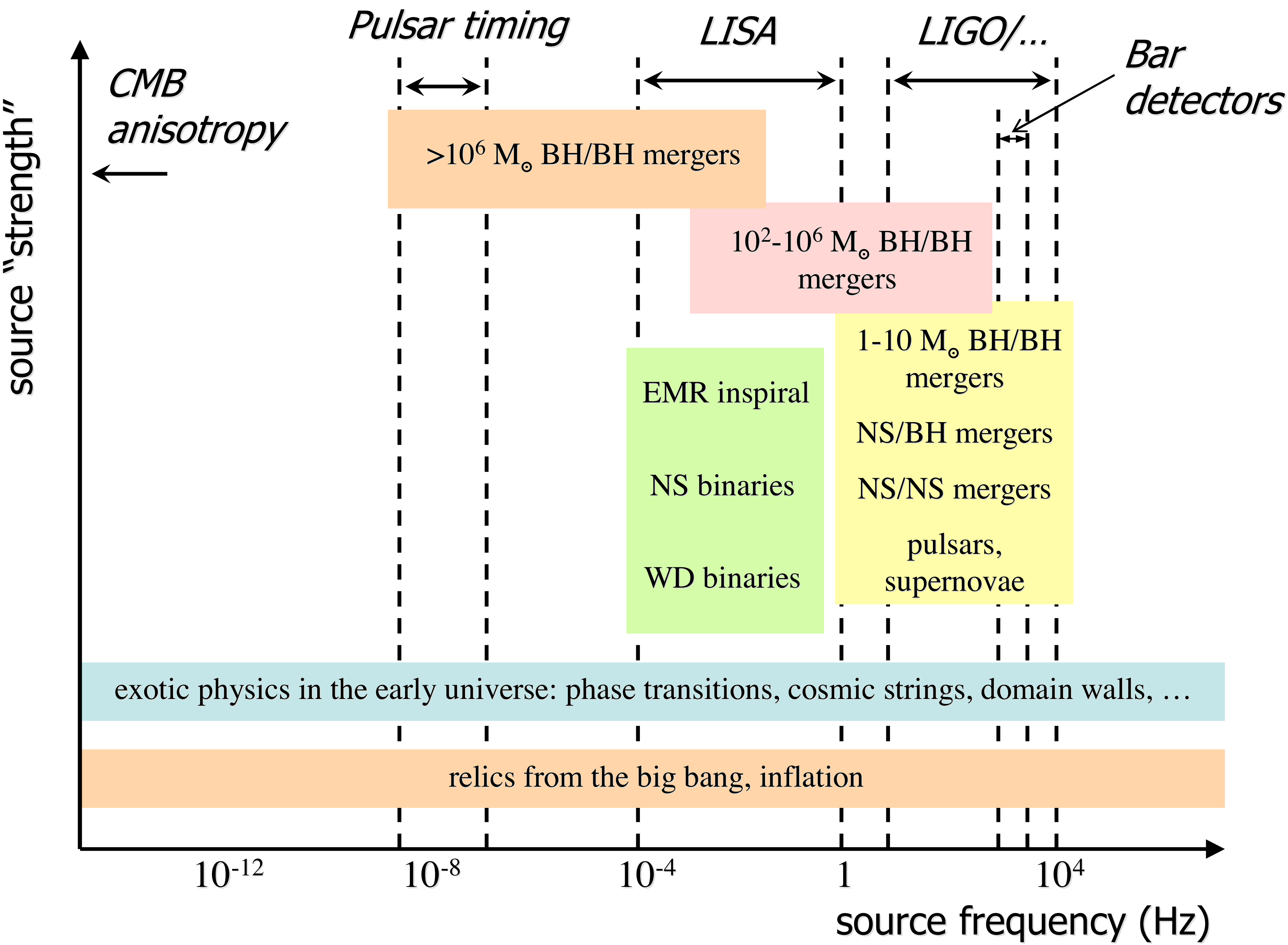


The Crab nebula … a supernovae remnant harboring a pulsar



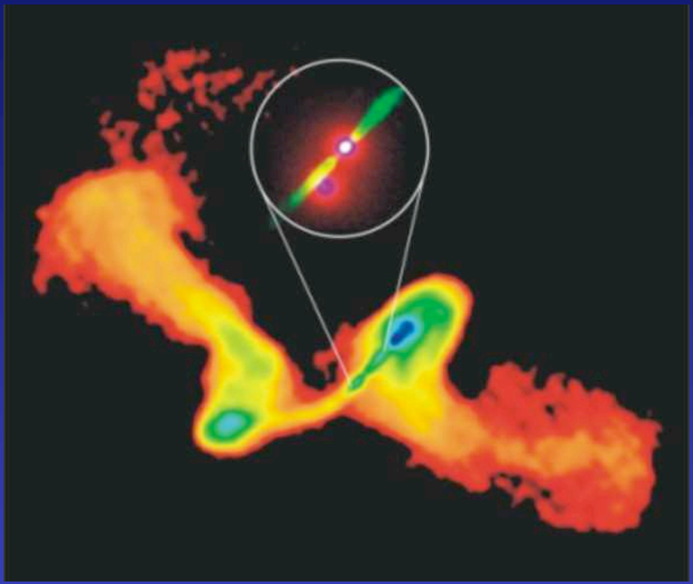
Segment of the CMB from WMAP

Overview of expected gravitational wave sources



Binary black holes in the Universe

- strong, though *circumstantial* evidence that black holes are ubiquitous objects in the universe
  - supermassive black holes (10^6 M$_\odot$ - 10^9 M$_\odot$) thought to exist at the centers of most galaxies
    - high stellar velocities near the centers of galaxies, jets in active galactic nuclei, x-ray emission, …
  - more massive stars are expected to form BH's at the end of their lives
    - a few dozen candidate stellar mass black holes in x-ray binary systems … companion too massive to be a neutron star



VLA image of the galaxy NGC 326, with HST image of jets inset. CREDIT: NRAO/AUI, STScI (inset)

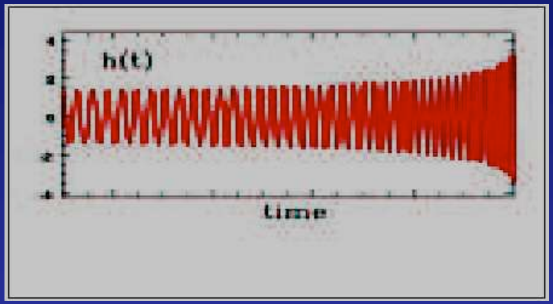
Two merging galaxies in Abell 400. Credits: X-ray, NASA/CXC/AIfA/D.Hudson & T.Reiprich et al.; Radio: NRAO/VLA/NRL)

- detection of gravitational waves from BH mergers would provide *direct* evidence for black holes, as well as give valuable information on stellar evolution theory and large scale structure formation and evolution in the universe
- this will also be an *unprecedented* test of general relativity, as the last stages of a merger takes place in the highly dynamical and non-linear strong-field regime

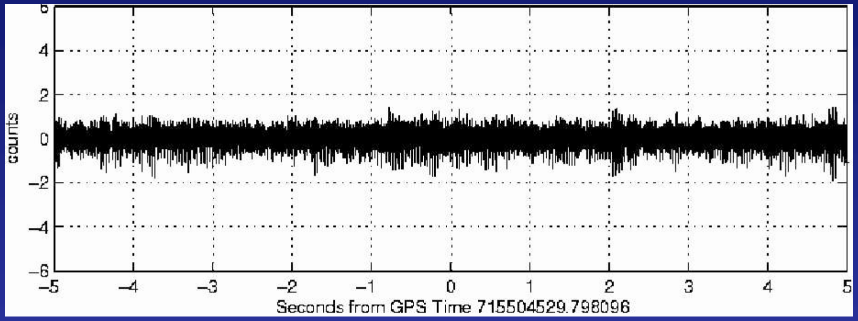
How do we observe sources?

- For the majority of sources, some knowledge of the nature of the source is required for detection of a signal
  - the gravitational wave strain is too small by the time the wave reaches earth to directly “see” the signal
- Matched filtering will be the primary tool for extracting small, quasi-periodic signals from the data stream
- Techniques such as the excess power method can be used for other sources, or if less is known about the exact nature of the source

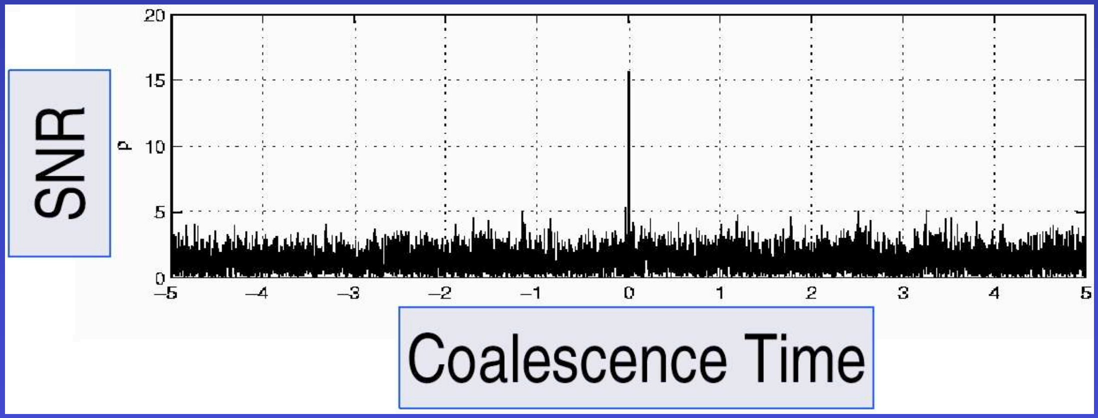
Matched filtering example



Simulated waveform from a binary black hole merger (M_1=M_2 ~ 10 M⊙, at ~ 15 Mpc)



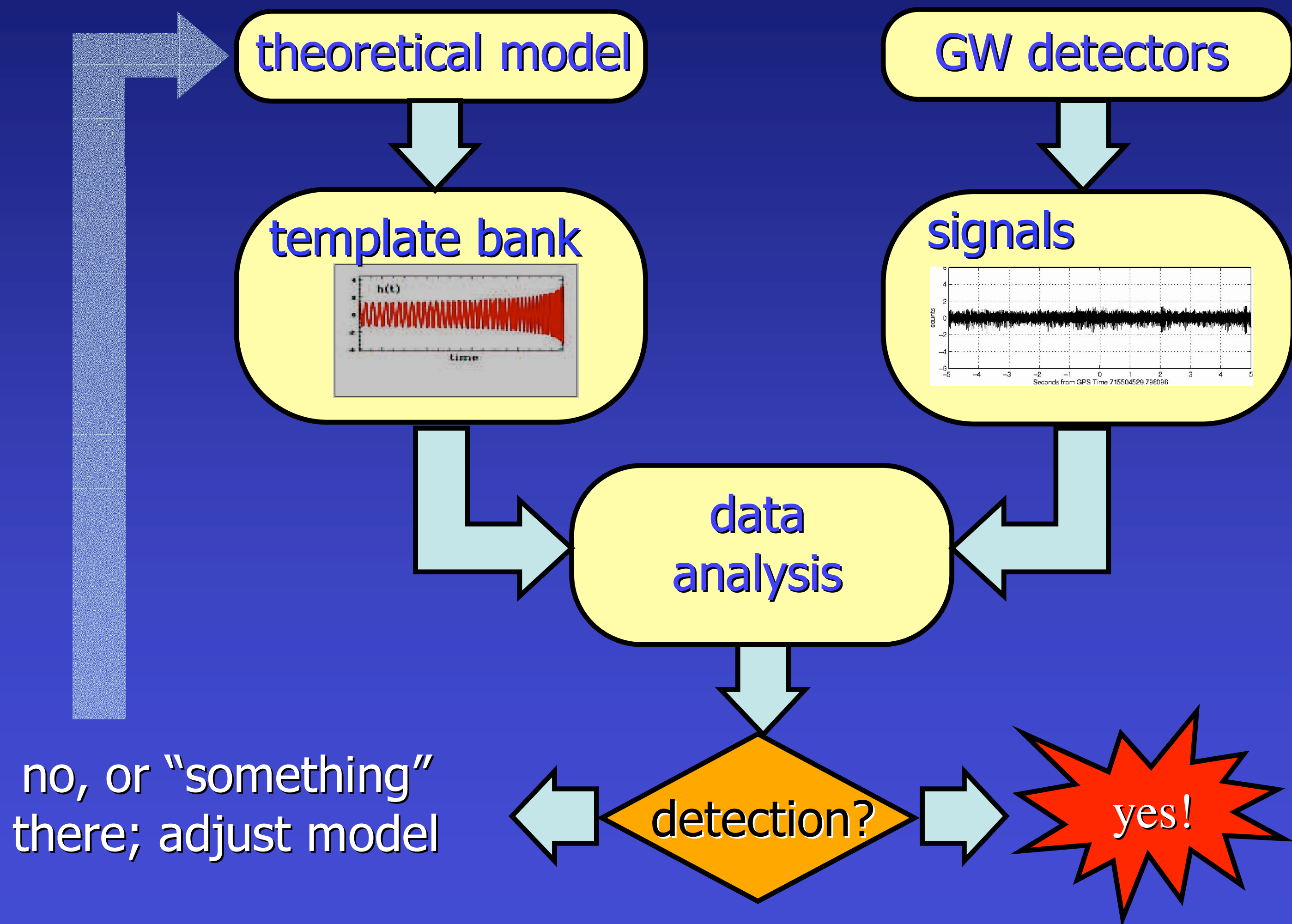
LIGO GW channel (as of ~ year ago) + injected waveform



Detection of the inspiral with a SNR~16 after application of the matched filtering algorithm

Images courtesy of Patrick Brady

Astronomy with gravitational waves



The need to better understand general relativity

- To take full advantage of the promise of gravitational wave detection, it is *imperative* that we understand the *non-linear, dynamical regime of general relativity*
  - current knowledge of astrophysically relevant solutions largely stems from patching together results about known stationary solutions, weak field calculations and global properties of the field equations
- Numerical methods can provide the tools to develop the required knowledge of general relativity

Numerical Relativity

- Numerical relativity is concerned with solving the field equations of general relativity

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

  using computers.

- When written in terms of the spacetime metric, defined by the usual line element

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

  the field equations form a *system of 10 coupled, non-linear, second order partial differential equations, each depending on the 4 spacetime coordinates*

  – it is this system of equations that we need to solve for the 10 metric elements (plus whatever matter we want to couple to gravity)

  – for many problems this has turned out to be quite an undertaking, due in part to the mathematical complexity of the equations, and also the heavy computational resources required to solve them

- The field equations may be complicated, but they are *the* equations that we believe govern the structure of space and time (barring quantum effects and ignoring matter). That they can, in principle, be solved in them in many "real-universe" scenarios is a remarkable and unique situation in physics.

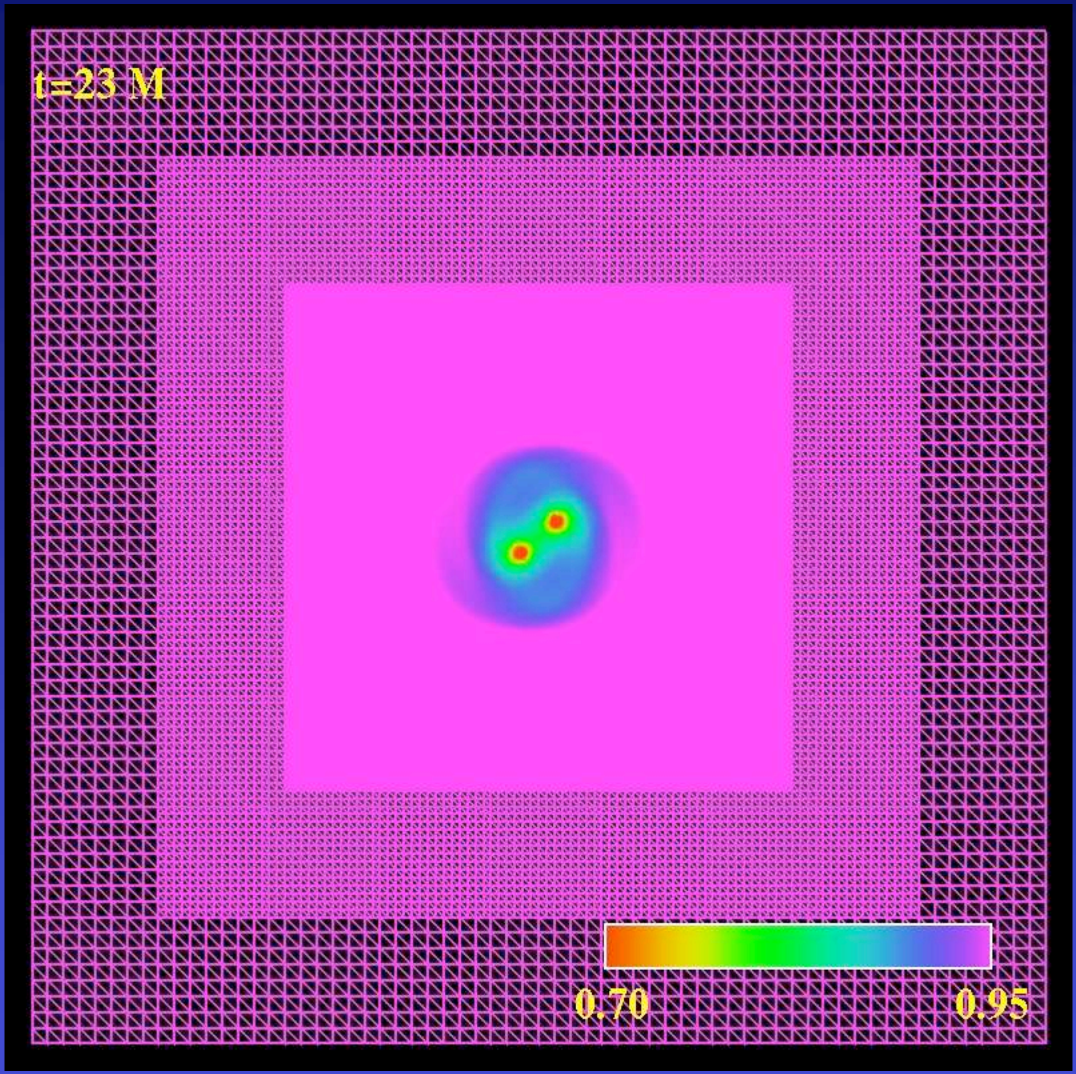
Computational challenges facing numerical relativity

- Each equation contains tens to hundreds of individual terms, requiring on the order of several thousand floating point operations *per grid point* with any evolution scheme.
- Problems of interest often have several orders of magnitude of relevant physical length scales that need to be well resolved. In an equal mass binary black hole merger for example:
  - radius of each black hole R~2M
  - orbital radius ~ 20M (which is also the dominant wavelength of radiation emitted)
  - outer boundary ~ 200M, as the waves must be measured in the weak-field regime to coincide with what detectors will see
- Can solve these problems with a combination of hardware technology — supercomputers — and software algorithms, in particular adaptive mesh refinement (AMR)

With the generalized 3D harmonic code, to perform 1 iteration of the equations at 1 grid point, need to evaluate ~ 5000 lines of code (4,500 of which are maple 'optimized' fortran).

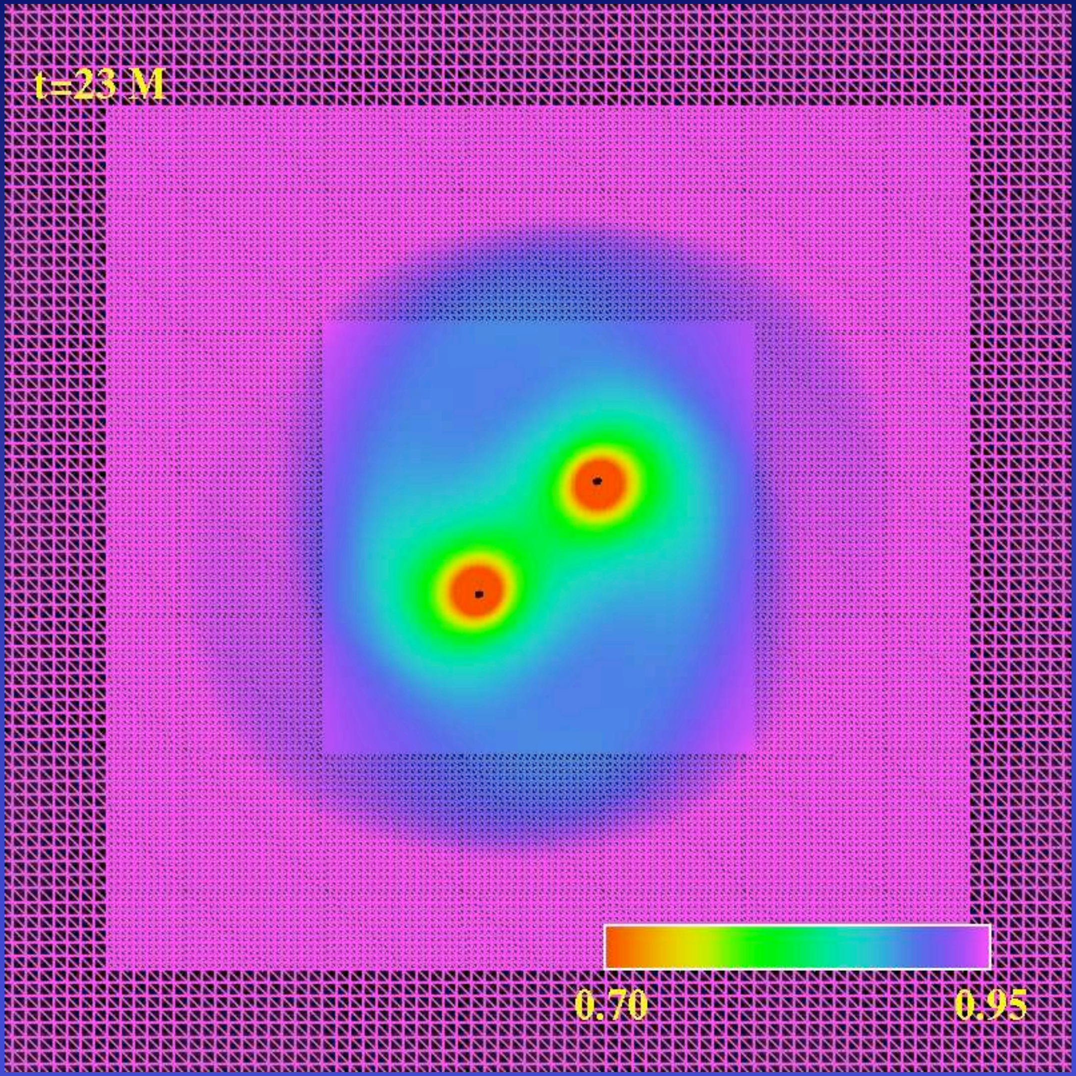
```
 t5875 = t5434*t5874
 t5876 = t5804*t1156
 t5878 = -t5873/2+t5875/2+t5876/2
 t5880 = t5704*gb_xy_z
 t5889 = gb_yy_x*t1156
 t5893 = 2*t1623-gb_yy_x+2*t1627
 t5904 = t333*t5872
 t5905 = t5484*gb_xz_y
 t5906 = gb_yz_x*t1156
 t5907 = t5906*t28
 t5908 = t5904-t5905+t5907
 t5911 = t2241*gb_ty_x
 t5923 = t5540*gb_xy_z
 t5924 = t333*t5874
 t5925 = t5906*t43
 t5926 = t5923-t5924+t5925
 t5928 = t2241*gb_tz_x
 t5931 = t1220*t5517*gb_xy_z/2
 t5942 = t2348*t1156
 t5944 = 0.3141592653589793D1*Hb_y0
 t5946 = t59*sny
 s2 = (t2241*gb_tt_x+t5804*t1202+(-t5510+t5511+t5513)*gb_tx_y+(t551
#6+t5519)*gb_tx_z+t5523+t5526-t5529+(-8*t5810*t5611-8*t5798*t5633)*
#phi1_t)*gbu_tz0+(2*t1493*gb_tx_x+t5434*t5668*gb_xx_t/2+2*t1171*t58
#23-4*t5648*phi1_y*t5827-8*t1388*t445-2*t5832+2*t1161*t1270)*gbu_xx
#0+(t5838*gb_tx_x+(-t5484*gb_xx_y+2*t5841+2*t5843)*gb_tx_y+t5718+t5
#793*t1505/2+t5721+2*t1730*t5849-8*t5681*t5398-16*t2939*t5454+2*t58
#56*t5858)*gbu_xy0
 s1 = s2+(2*t1216*t4838+(-t333*t5668+2*t5866+2*t5868)*gb_tx_z+t5736
#+t5878*gb_xx_t+t5738+2*t1730*t5880-8*t5709*t5398-16*t2939*t5479)*g
#bu_xz0+(t5889*t459+t5893*gb_ty_x+t5793*t1580/2-4*t5723*t5398-8*t55
#03*t1794*t5363)*gbu_yy0+(t5908*gb_tx_y+t5889*t598+t5911+t5893*gb_t
#z_x+t5878*gb_xy_t+t5793*t1838/2-8*t5746*t5398-16*t1388*t3764*t5535
#)*gbu_yz0+(t5926*gb_tx_z+t5928-t5931+(t5875/2+t5876/2)*gb_xz_t-4*t
#5759*t5398-8*t5557*t1878*t5363)*gbu_zz0
 t5970 = s1+t5942*gb_tx_x+(-t5944*t28/4+t5946*csy/4)*t5363*gb_xx_t-
#t1616*t5719/4+t1616*t5743/2+16*t5381*phi1_y*t5827+(t5944*t167*t28-
#t5946*t167*csy+t5566*t1676/2)*t5372-gb_xx_ty*0.3141592653589793D1*
#t5488/2
 t5973 = 0.3141592653589793D1*gb_zz_x
 t5982 = 2*t1524
 t5987 = t1173*gb_xz_x
 t5991 = t5517*phi1_t
 t5996 = t1161*gb_xz0
 t6005 = t1190*gb_xz_x
```

Mesh refinement example



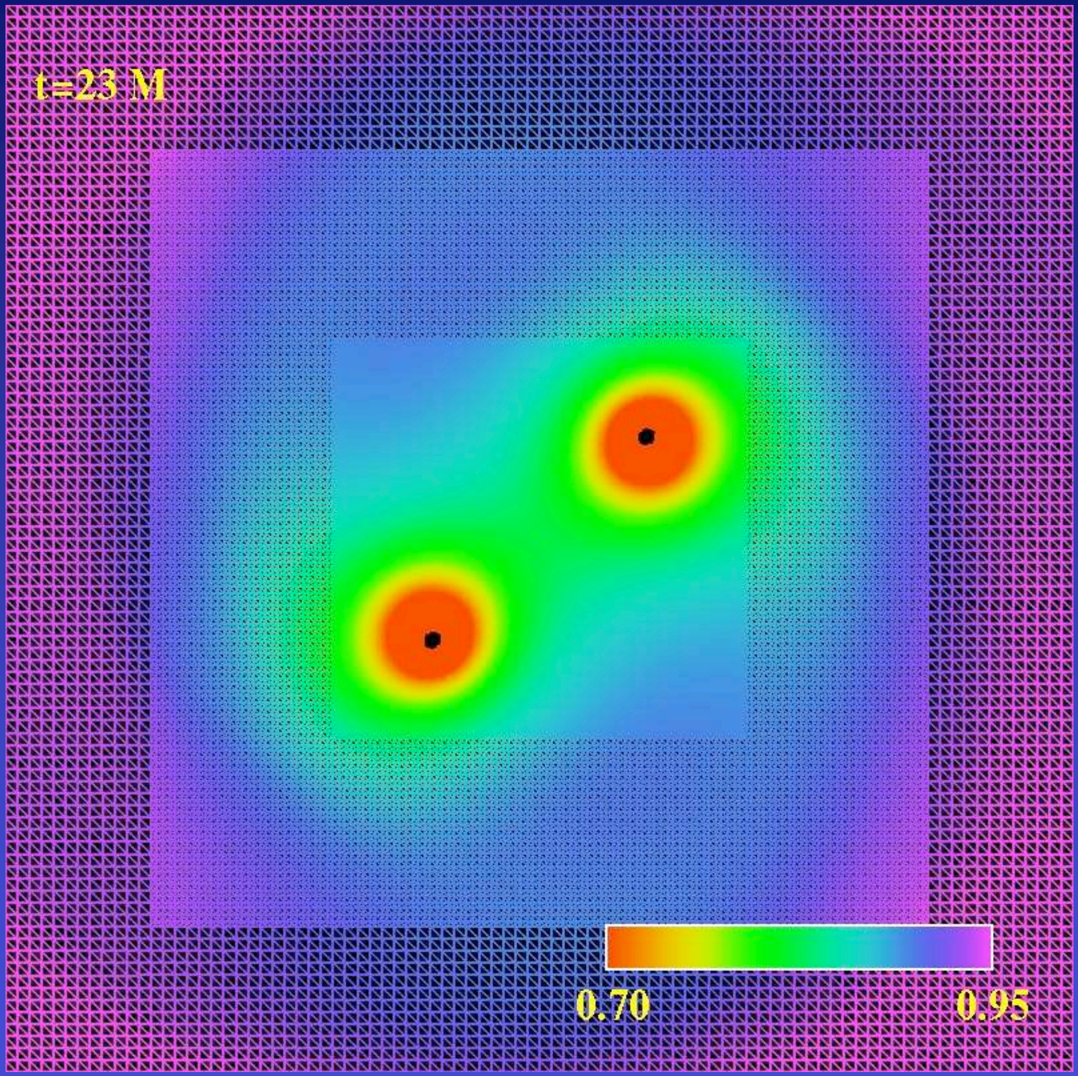
Lapse α, $z=0$ slice

Mesh refinement example



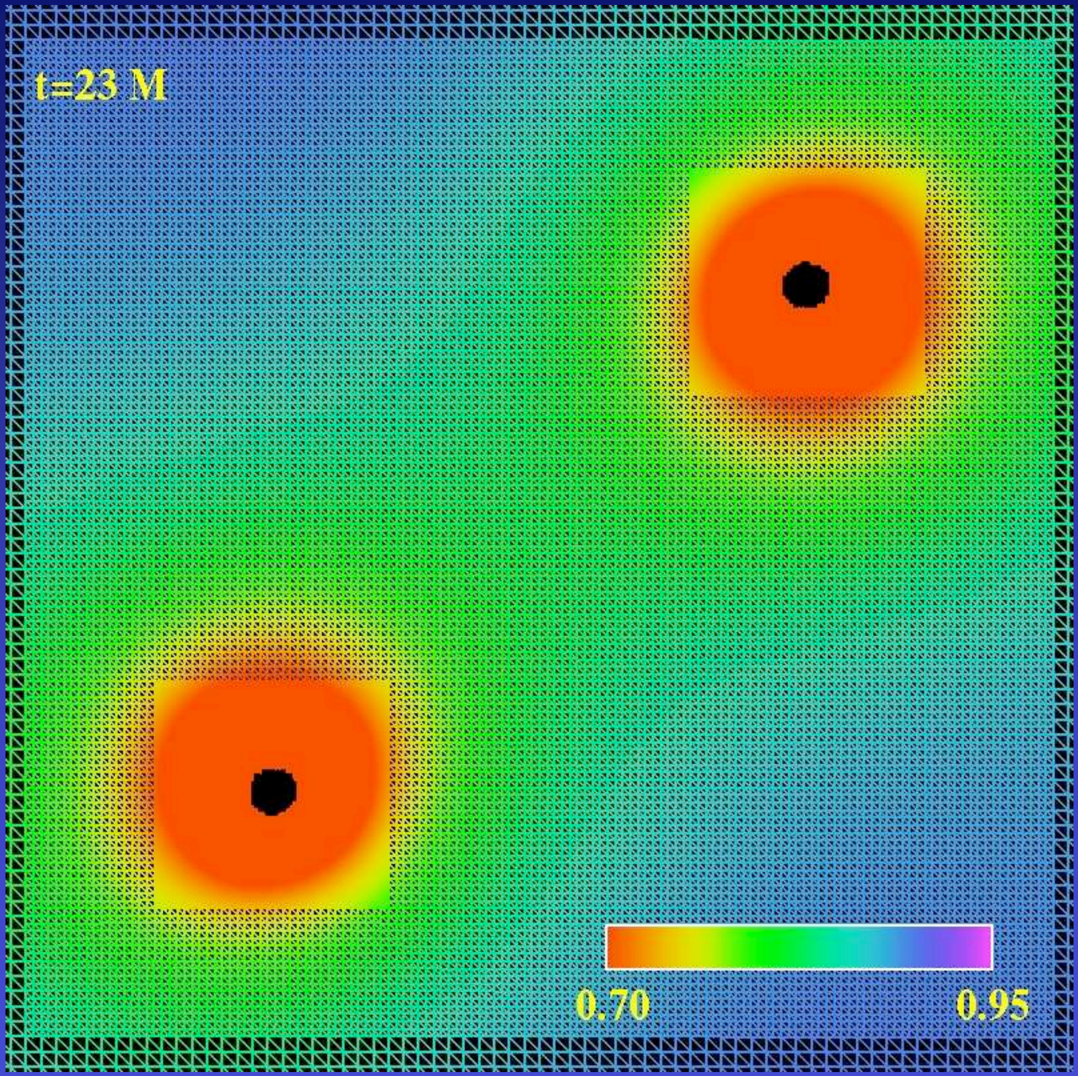
Lapse α, $z=0$ slice

Mesh refinement example



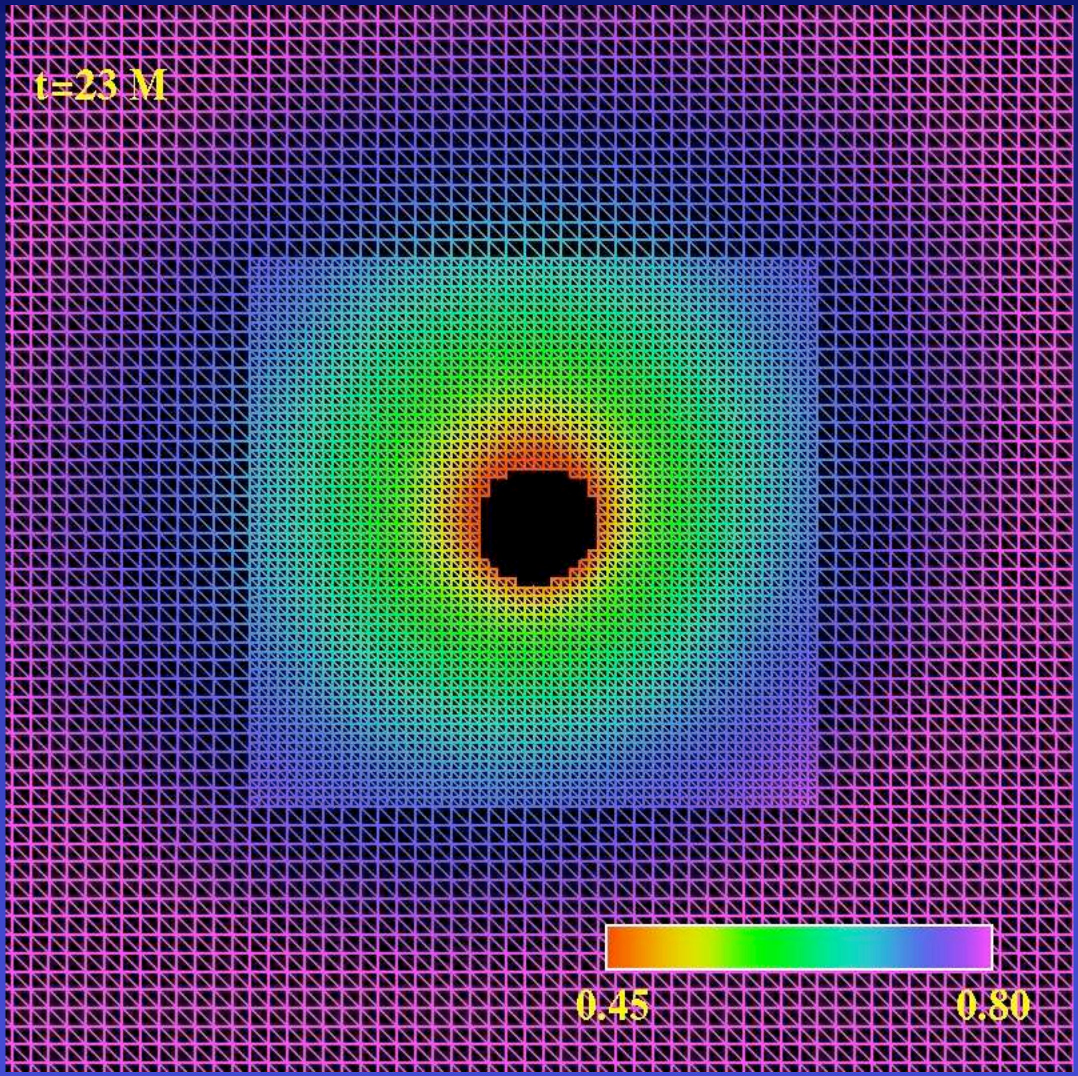
Lapse α , $z=0$ slice

Mesh refinement example



Lapse α, $z=0$ slice

Mesh refinement example



Lapse α, $z=0$ slice, NOTE change of color scale

Mathematical challenges facing numerical relativity

- As mentioned before, when written out in a coordinate bases in a their usual form the field equations are a system of 10 coupled, non-linear, second order partial differential equations
- Due to the covariance of the theory, a unique solution cannot be found prior to choosing the 4 degrees of coordinate freedom — *the system of equations is under-determined*
  - moreover, the character (hyperbolic, parabolic, elliptic) of the differential equations is largely undetermined until appropriate coordinate choices are made
- Posed as an initial value problem, 4 of the field equations only involve the initial data at any given constant time slice; i.e. they are constraint equations — *the system of equations is over-determined*

Minimal requirements for a formulation of the field equations that *might* form the basis for a successful numerical integration scheme

- Choose coordinates/system-of-variables that fix the character of the equations
  - three common choices
    - free evolution — system of hyperbolic equations
    - constrained evolution — system of hyperbolic and elliptic equations
    - characteristic or null evolution — integration along the lightcones of the spacetime
- For free evolution, need a system of equations that is well behaved off the *"constraint manifold"*
  - analytically, if satisfied at the initial time the constraint equations of GR will be satisfied for all time
  - numerically the constraints can only be satisfied to within the truncation error of the numerical scheme, hence we do not want a formulation that is "unstable" when the evolution proceeds slightly off the constraint manifold
- Need well behaved coordinates (or gauges) that do not develop pathologies when the spacetime is evolved
  - typically need dynamical coordinate conditions that can adapt to unfolding features of the spacetime
- Boundary conditions also historically a source of headaches
  - naive BC's don't preserve the constraint nor are representative of the physics
  - fancy BC's can preserve the constraints, but again miss the physics
  - solution … compactify to infinity
- Geometric singularities in black hole spacetimes need to be dealt with

Numerical relativity using generalized harmonic coordinates – a brief overview

- *Harmonic coordinates*

$$\nabla^\alpha \nabla_\alpha x^\mu \equiv \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g}\, g^{\alpha\mu} \right) = 0$$

  does to the Einstein equations what the Lorenz gauge does to Maxwell's equations … the *principle part* of each component of the Einstein tensor becomes a wave equation for the corresponding metric element

$$\nabla^\delta \nabla_\delta g_{\alpha\beta} + \ldots = 0$$

  - the character of each field equation is now hyperbolic
  - the ellipsis denote all the lower order terms, which contain the non-linearity and messy couplings between the metric elements

- Harmonic coordinates are in a sense older than the field equations themselves, as they were used by Einstein as early as 1912 while searching for a relativistic theory of gravity

- over the years they have played an instrumental role in the formal analysis of the field equations, and the study of gravitational radiation

  - "avoided" in numerical relativity because of the *somewhat* misguided belief that they were prone to developing coordinate pathologies in generic scenarios

  - Garfinkle *[PRD 65, 044029 (2002)]* recently noted a possible resolution to this problem

Generalized Harmonic Coordinates

- Generalized harmonic coordinates introduce a set of arbitrary *source functions H^{μ}* into the usual definition of harmonic coordinates

$$\nabla^{\alpha}\nabla_{\alpha}x^{\mu} \equiv \frac{1}{\sqrt{-g}}\partial_{\alpha}\left(\sqrt{-g}\, g^{\alpha\mu}\right) = H^{\mu}$$

  - note that *any* metric in *any* coordinate system can be viewed as a generalized harmonic metric

- If we now treat the *H^{μ} as independent functions*, we can still write the field equations in the desirable wave-like form of harmonic coordinates:

$$g^{\gamma\delta} g_{\alpha\beta,\gamma\delta} + 2 g^{\gamma\delta}{}_{,(\alpha} g_{\beta)\delta,\gamma} + 2H_{(\alpha,\beta)} - 2H_{\delta}\Gamma^{\delta}_{\alpha\beta} + 2\Gamma^{\gamma}_{\delta\beta}\Gamma^{\delta}_{\gamma\alpha} + 8\pi\left(2T_{\alpha\beta} - g_{\alpha\beta}T\right) = 0$$

where

$$\Gamma^{\delta}_{\alpha\beta} \equiv \frac{1}{2} g^{\delta\varepsilon}\left(g_{\alpha\varepsilon,\beta} + g_{\beta\varepsilon,\alpha} - g_{\alpha\beta,\varepsilon}\right)$$

- The source functions now encode the coordinate freedom in general relativity, and to close the system we must specify some additional equations for the H^{μ}

Constraint damping

- Note that we still have constraints in the generalized harmonic scheme that need to be solved at the initial time for self-consistent solutions:

$$\boxed{C^{\mu} \equiv H^{\mu} - \nabla^{\alpha}\nabla_{\alpha}x^{\mu}}$$

- It turns out that free evolution of spacetimes containing black holes with "plain" harmonic evolution does not have desirable evolution properties if the numerical data contains small violations of the constraints

- The (apparent) cure, as suggestion by C. Gundlach et al (*[C. Gundlach, J. M. Martin-Garcia, G. Calabrese, I. Hinder, gr-qc/0504114]* based on earlier work by Brodbeck et al *[J. Math. Phys. 40, 909 (1999)]*) is to modify the Einstein equations in harmonic form as follows:

$$\boxed{g^{\alpha\beta}g_{\mu\nu,\alpha\beta} + \ldots + \kappa\left(n_{\mu}C_{\nu} + n_{\nu}C_{\mu} - g_{\mu\nu}n^{\alpha}C_{\alpha}\right) = 0}$$

$n_u=-\alpha\partial_u t$ is a unit timelike vector normal to $t=const.$ hypersurfaces, with proper time measured by an observer moving along n_u given by the *lapse function* α, and κ is a constant parameter

  - note that any solution to the field equations must have $C^u=0$, so we are adding "nothing" to them!
  - however, if C^u happens to be non-zero but small, and the spacetime being evolved is a small perturbation of Minkowski space, Gundlach et al showed that this modification causes all finite wavelength components of C^u to be exponentially damped with time
    - no proof that this damping property extends to more general scenarios, but the numerical success in binary black hole simulations suggest it may

Summary of Equations solved

- Einstein equations in generalized harmonic form with constraint damping:

$$g^{\gamma\delta} g_{\alpha\beta,\gamma\delta} + 2 g^{\gamma\delta}{}_{,(\alpha} g_{\beta)\delta,\gamma} + 2H_{(\alpha,\beta)} - 2H_{\delta}\Gamma^{\delta}_{\alpha\beta} + 2\Gamma^{\gamma}_{\delta\beta}\Gamma^{\delta}_{\gamma\alpha} + 8\pi\left(2T_{\alpha\beta} - g_{\alpha\beta}T\right)$$
$$+ \kappa\left(n_\mu C_\nu + n_\nu C_\mu - g_{\mu\nu} n^\alpha C_\alpha\right) = 0$$

- Gauge evolution equations

$$\nabla^\mu \nabla_\mu H_t = -\xi_1 \frac{\alpha - 1}{\alpha^n} + \xi_2 \partial_\mu H_t \cdot n^\mu$$
$$H_x = H_y = H_z = 0$$

  – time source function prevents the lapse from "collapsing" in black hole spacetimes

- Matter stress energy supplied by a massless scalar field Φ:

$$T_{\alpha\beta} = 2\Phi_{,\alpha}\Phi_{,\beta} - g_{\alpha\beta}\Phi_{,\mu}\Phi^{,\mu}$$
$$\nabla^\mu \nabla_\mu \Phi = 0$$

The two body problem

- Newtonian gravity solution for the dynamics of two point-like masses in a bound orbit: motion along an ellipse

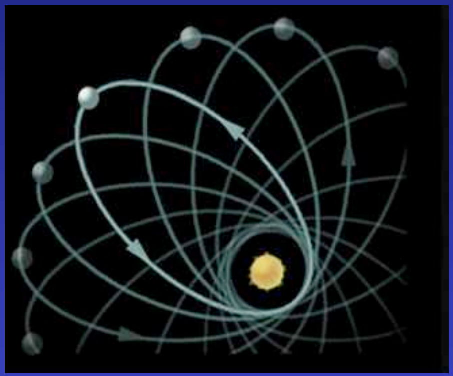
- in general relativity there is no (analytic) solution … several approximations with different realms of validity

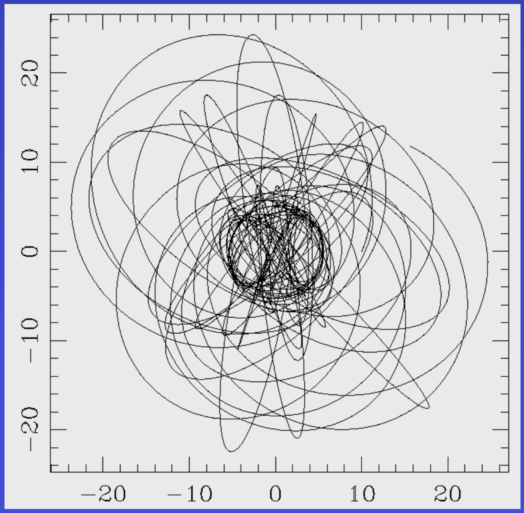
  - test particle limit
    - geodesic motion of a particle about a black hole (i.e. self-gravity of particle is ignored)
    - already get some very interesting behavior
      - perihelion precession
      - unstable and chaotic orbits
      - "zoom-whirl" behavior

  - Post-Newtonian (PN) expansions
    - self-gravity accounted for, though slow motion (relative to c) and weak gravitational fields assumed
    - begins to incorporate "radiation-reaction"; i.e. how the orbit decays via the emission of gravitational waves

  - black hole (BH) perturbation theory
    - can be used to model the "ring-down" of the final BH that is formed in a collision
    - can also describe the radiation caused by a test particle in orbit about the BH

- binary black hole mergers
  - all the above assumptions break down close to the merger of comparable mass BHs: self gravity can't be ignored, the gravitational fields are not weak, and the BHs are moving at sizeable fractions of the speed of light





From N. Cornish and J. Levin, CQG 20, 1649 (2003)

Brief (and incomplete) history of the binary black hole problem in numerical relativity

- L. Smarr, *PhD Thesis (1977)* : First head-on collision simulation
- P. Anninos, D. Hobill, E.Seidel, L. Smarr, W. Suen *PRL 71, 2851 (1993)* : Improved simulation of head-on collision
- B. Bruegmann *Int. J. Mod. Phys. D8, 85 (1999)* : First grazing collision of two black holes
- B. Bruegmann, W. Tichy, N. Jansen *PRL 92, 211101 (2004)* : First full orbit of a quasi-circular binary
- FP, *PRL 95, 121101 (2005)* : First "complete" simulation of a non head-on merger event: orbit, coalescence, ringdown and gravitational wave extraction
- M. Campanelli, C. O. Lousto, P. Marronetti, Y. Zlochower *(gr-qc/0511048);* J. G. Baker, J. Centrella, D. Choi, M. Koppitz, J. van Meter *(gr-qc/0511103)* : similar complete merger event as FP(2005), though using very different numerical techniques. There methods reproduced by F. Herrmann, D. Shoemaker, P. Laguna *(gr-qc/0601026).*

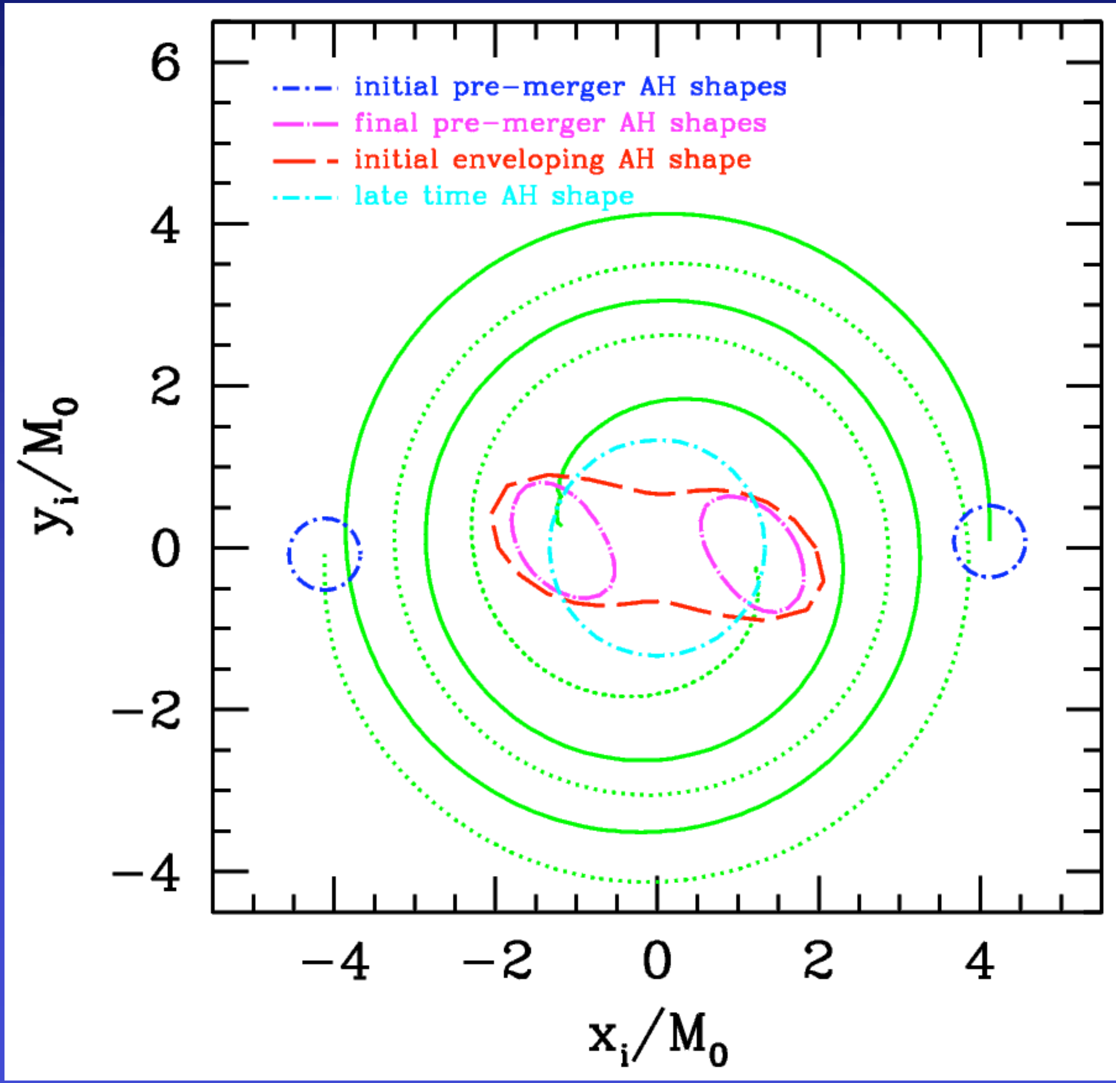
The initial data problem …

- In is *not* easy specifying astrophysically realistic binary black hole (BBH) initial data for evolution
  - the initial geometry must satisfy the constraint equations, and so cannot be freely specified
    - state of the art methods available today for solving the constraints for BBH initial data do *not* include the radiation that would have been generated by the prior inspiral history of the BHs
    - PN (and other approximate solutions) do *not* satisfy the constraints, and might not even have black holes
      - several suggestions for melding PN methods with constraint equation solving methods, though none have yet been tested

Evolution of Cook-Pfeiffer Quasi-Circular Initial Data Sets

- Initial data provided by H. Pfeiffer, based on solutions to the constraint equations with free data and black hole boundary conditions as described in *Cook and Pfeiffer, PRD 70, 104016 (2004)*
  - approximation to the structure of spacetime describing a BBH system composed of equal mass, corotating black holes, initially on circular orbits
  - "good" assumptions used, except
    - no gravitational radiation content
    - no tidal deformation of the BHs
    - no radial component to BH velocities
  - data sets are parameterized by the initial separation of the binaries
    - the closer the BHs are the more pronounced the above errors will be

A Cook-Pfeiffer Inspiral Orbit



Initial coordinate (proper) separation:

7.4M (9.8M)

Final BH angular momentum:

$J=0.70 \pm 0.02\ M^2$

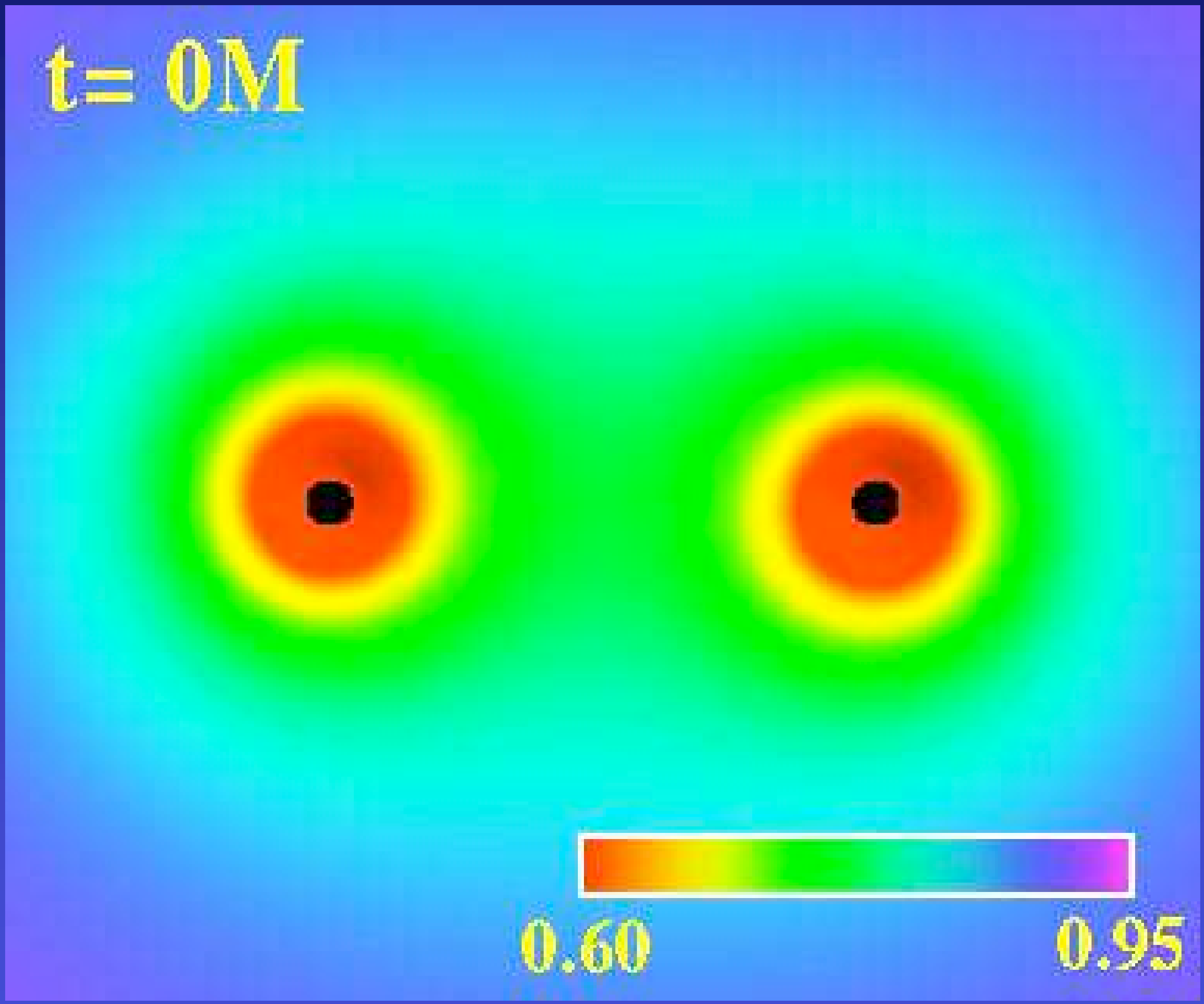
Energy radiated:

$0.043M \pm 0.004M$

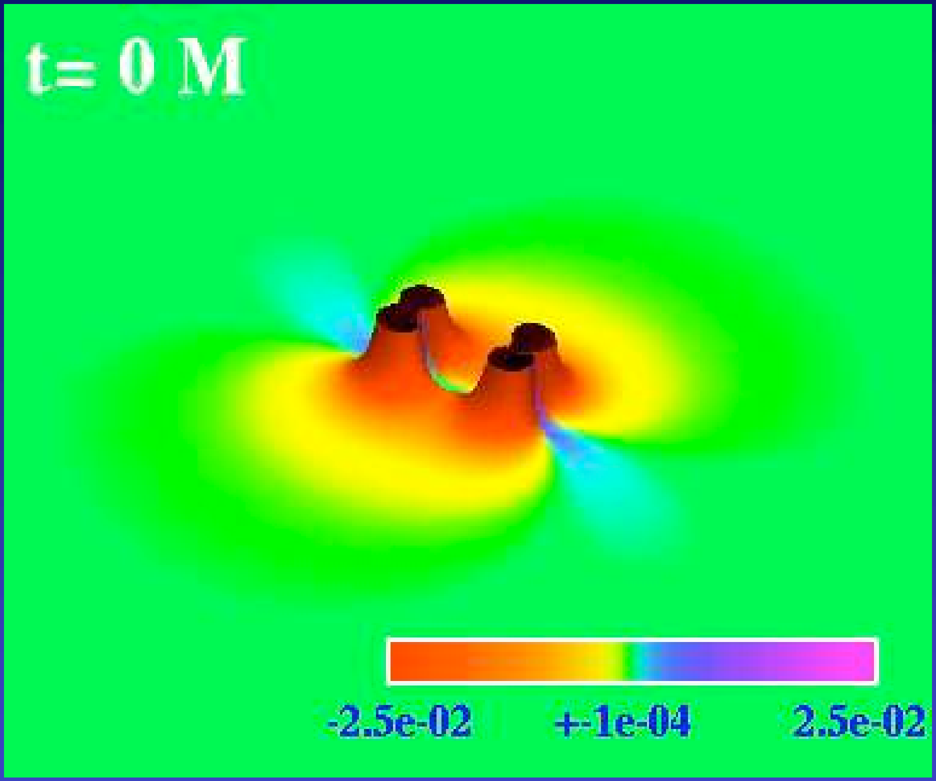
Errors estimated from simulations with three characteristic resolutions.

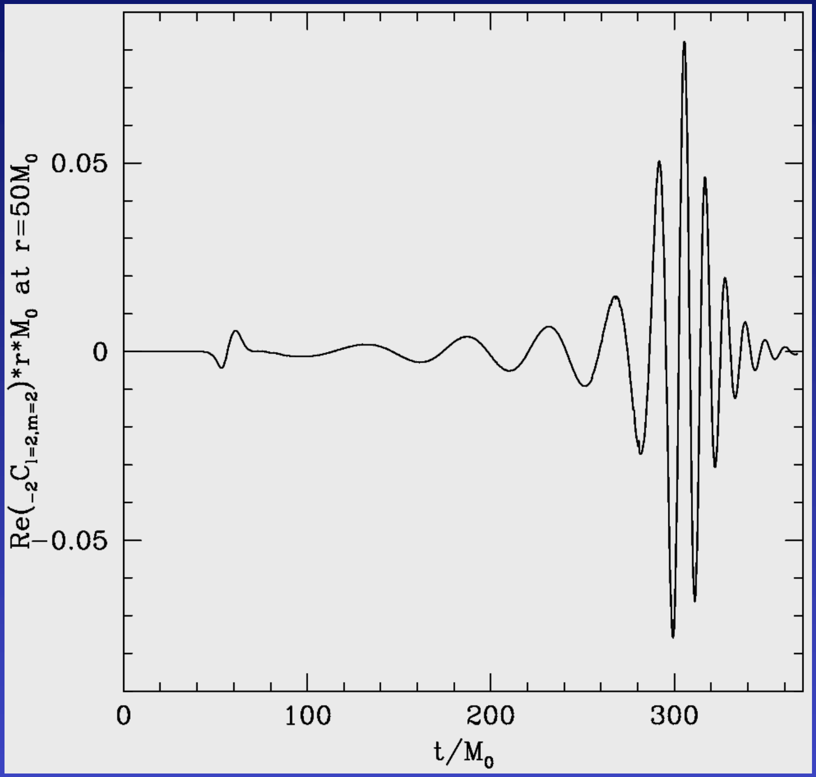
Highest-res simulation details: ~ 60,000 CPU hours on UT *lonestar* cluster (3 weeks total on 128 nodes), ~ 2TB disk usage (infrequent output) , ~25GB total RAM usage. (other machines used include Westgrid's *glacier* and *matrix*, and UBC's *vnp4*)

Lapse function α, orbital plane



Gravitational waves





Real component of the Newman-Penrose scalar Ψ_4(times rM), orbital plane. Here, color and height of the surface represents the magnitude of Ψ_4. Far from the source the real and imaginary components of Ψ_4 are just the second time derivatives of the "plus" and "cross" polarizations of the gravitational wave.

The real component of the spin -2 weight, l=2, m=2 spherical harmonic component of Ψ_4 times rM , measured at a coordinate distance of $50M$ from the center of the orbit.

Convergence tests suggest dominant source of error is a near-linear drift in the phase of the waveform until a common horizon forms, which occurs ~ $20M$ *before* the peak in amplitude.

Total phase error: $\pm 0.13\ 2\pi$

What does this wave represent?

- Scale the system to two 10 solar mass (~ 2×10^{31} kg) BHs
  - radius of each black hole in the binary is ~ 30km
  - radius of final black hole is ~ 60km
  - distance from the final black hole where the wave was measured ~ 1500km
  - frequency of the wave ~ 200Hz (early inspiral) - 800Hz (ring-down)
  - fractional oscillatory "distortion" in space induced by the wave transverse to the direction of propagation has a *maximum* amplitude $\Delta L/L \sim 3 \times 10^{-3}$
    - a 2m tall person will get stretched/squeezed by ~ 6 mm as the wave passes
    - LIGO's arm length would change by ~ 12m. Wave amplitude decays like 1/distance from source; e.g. at 10Mpc the change in arms ~ 5×10^{-17}m (1/20 the radius of a proton, which is well within the ballpark of what LIGO is trying to measure!!)
  - despite the seemingly small amplitude for the wave, the energy it carries is enormous — around 10^{30} kg c^2 ~ 10^{47} J ~ 10^{54} ergs (peak luminosity is about 1/100th the Planck luminosity of 10^{59}ergs/s !!)

Brief aside: scalar fields

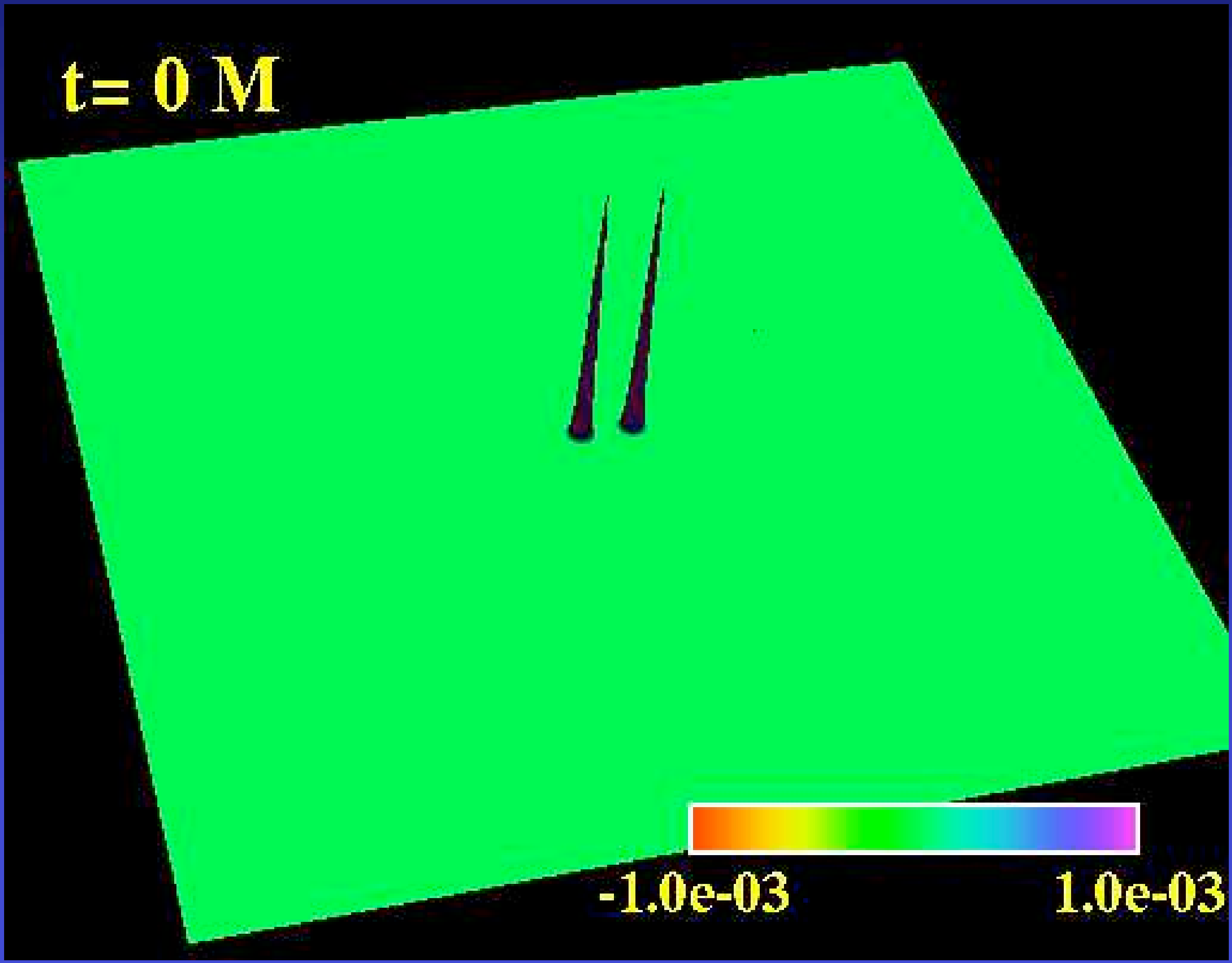
- Do scalar fields play a role in the universe?
  - Higgs field?
  - Dark matter, dark energy candidates?
  - Responsible for driving inflation?
- Why do we use scalar fields?
  - simple form of matter
    - very much like electromagnetic radiation without polarization
    - free of shocks, discontinuities, turbulence, multiple sounds speeds, etc. found in "nastier" forms of matter
  - therefore, ideal as a source to drive strong field dynamics in general relativity

Scalar field collapse driven binaries

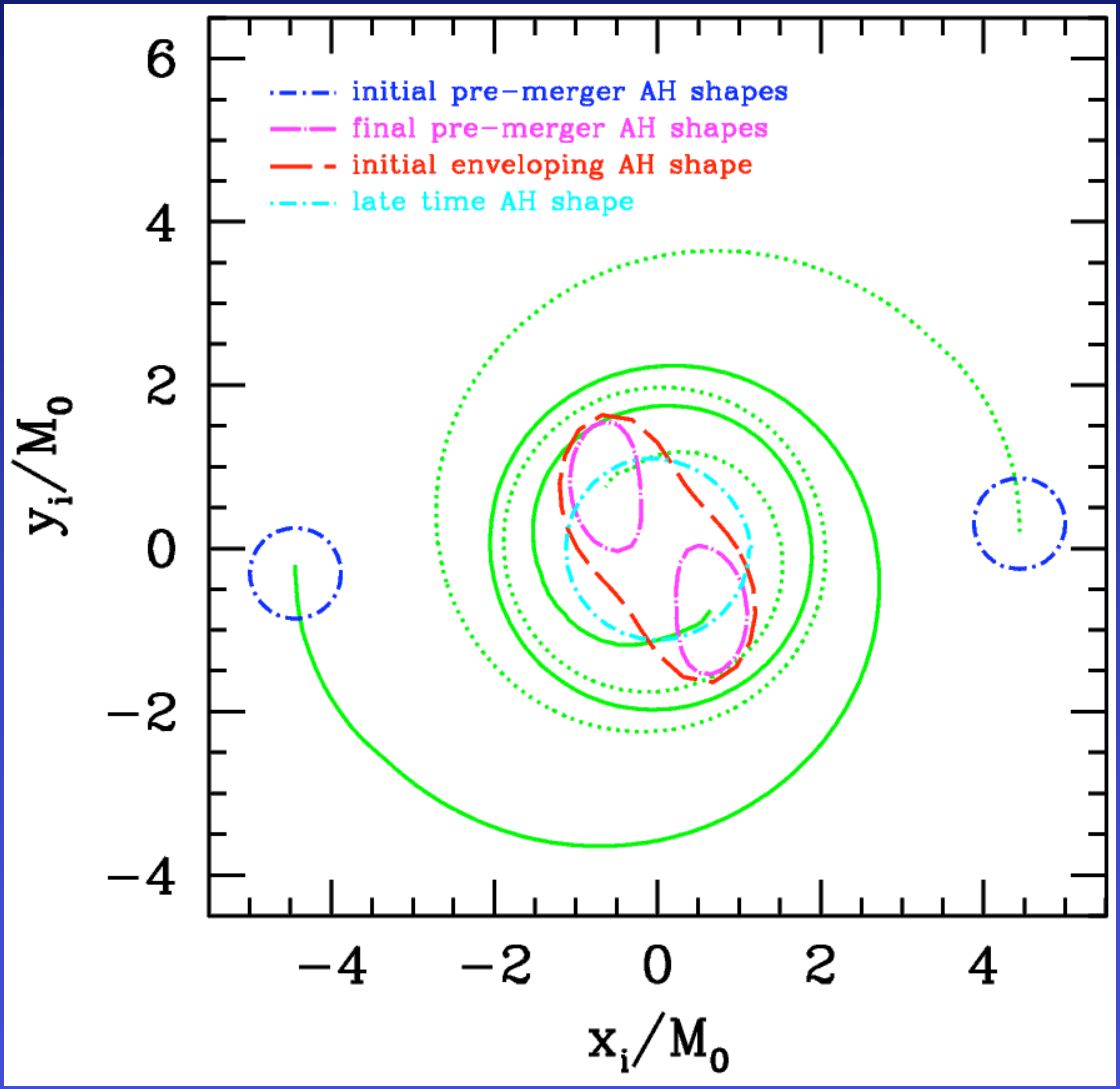
- Look at equal mass mergers
  - initial scalar field pulses separated a coordinate (proper) distance $8.9M$ ($10.8M$) on the x-axis, one boosted by v in the $+y$ direction, the other by v in the $-y$ direction
    - note, resultant black hole velocities are related to, but not equal to v
- To find interesting orbital dynamics, tune the parameter v to get as many orbits as possible
  - in the limit as v goes to 0, get head-on collisions
  - in the large v limit, black holes are deflected but fly apart
- Generically these black hole binaries will have some eccentricity (not easy to define given how close they are initially), and so arguably of less astrophysical significance
  - want to explore the non-linear interaction of BH's in full general relativity

Scalar field $\phi.r$, compactified (code) coordinates

$$\bar{x} = \tan(x\pi/2), \bar{y} = \tan(y\pi/2), \bar{z} = \tan(z\pi/2)$$



Sample Orbit



h-resolution runs

v	n	p_m/M_0	d_m/M_0	m_f/M_0	a/m_f	(E/M_0)
0.21000	1.3	-	-	0.89 ± 0.03	0.75 ± 0.05	0.032
0.21125	1.4	-	-	0.88 ± 0.03	0.74 ± 0.05	0.035
0.21234	2.3	-	-	0.83 ± 0.03	0.73 ± 0.05	?
0.21250	2.7	4.0	3.6	–	–	0.020
0.21500	1.5	5.5	4.6	–	–	0.006
0.22000	1.0	7.2	5.8	–	–	0.005

6/8 h-resolution runs

v	n	p_m/M_0	d_m/M_0	m_f/M_0	a/m_f	(E/M_0)
0.20960	0.9	–	–	0.97 ± 0.01	0.65 ± 0.03	0.028
0.21750	1.4	–	–	0.92 ± 0.01	0.72 ± 0.03	0.037
0.21875	2.0	–	–	0.88 ± 0.01	0.70 ± 0.03	0.046
0.21906	2.4	–	–	0.86 ± 0.01	0.70 ± 0.03	0.052
0.219180	2.8	-	-	0.82 ± 0.02	0.70 ± 0.05	0.063
0.219200	3.0	-	-	0.80 ± 0.02	0.75 ± 0.05	0.064
0.219209	3.3	-	-	0.78 ± 0.02	0.71 ± 0.05	0.067
0.219214	3.7	-	-	0.75 ± 0.02	0.71 ± 0.05	0.074
0.219219	4.9	3.2	3.0	–	–	0.058
0.21938	2.5	4.8	4.2	–	–	0.019
0.22000	1.9	5.3	4.4	–	–	0.014

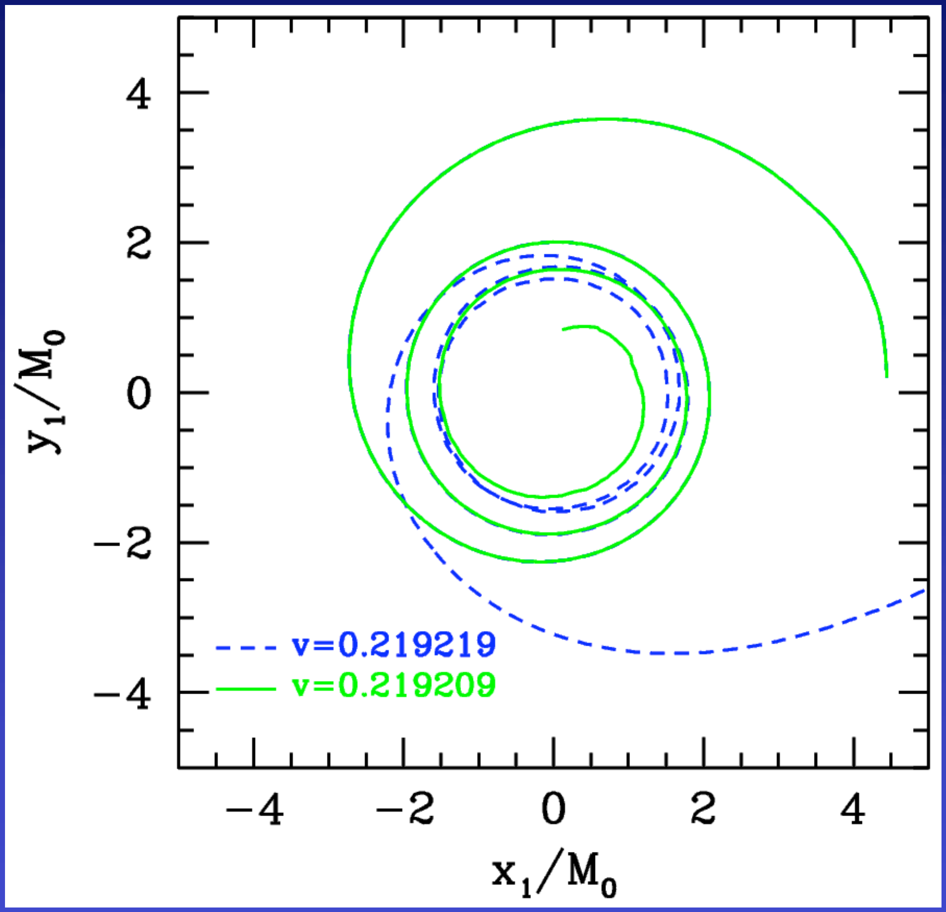
4/8 h-resolution runs

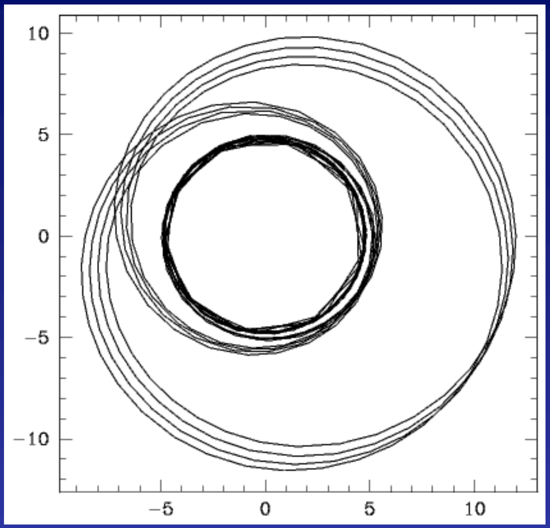
v	n	p_m/M_0	d_m/M_0	m_f/M_0	a/m_f	(E/M_0)
0.21500	1.4	–	–	0.945 ± 0.005	0.71 ± 0.02	0.042
0.22000	2.1	5.7	4.8	–	–	?

Early indications of "extreme" sensitivity to initial conditions

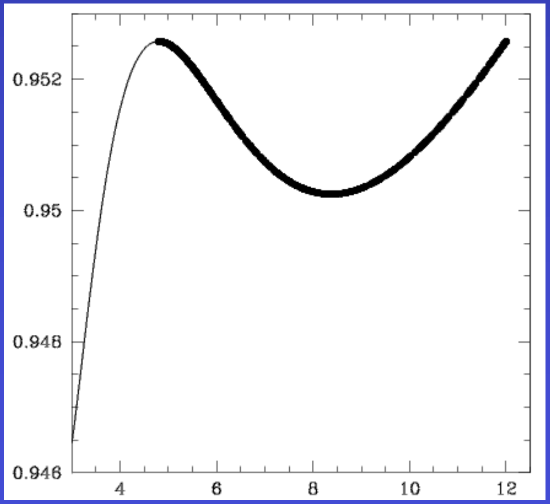
- What's going on??
  - *warning:* large cumulative numerical errors, especially for the lower resolutions (though does not necessarily mean qualitative features are wrong, c.f. critical gravitational collapse)
  - could be the fully non-linear analogue of "zoom-whirl" behavior in test particle orbits

Orbits





example of a homoclinic particle orbit it Schwarzschild (above) and the corresponding effective potential (below)

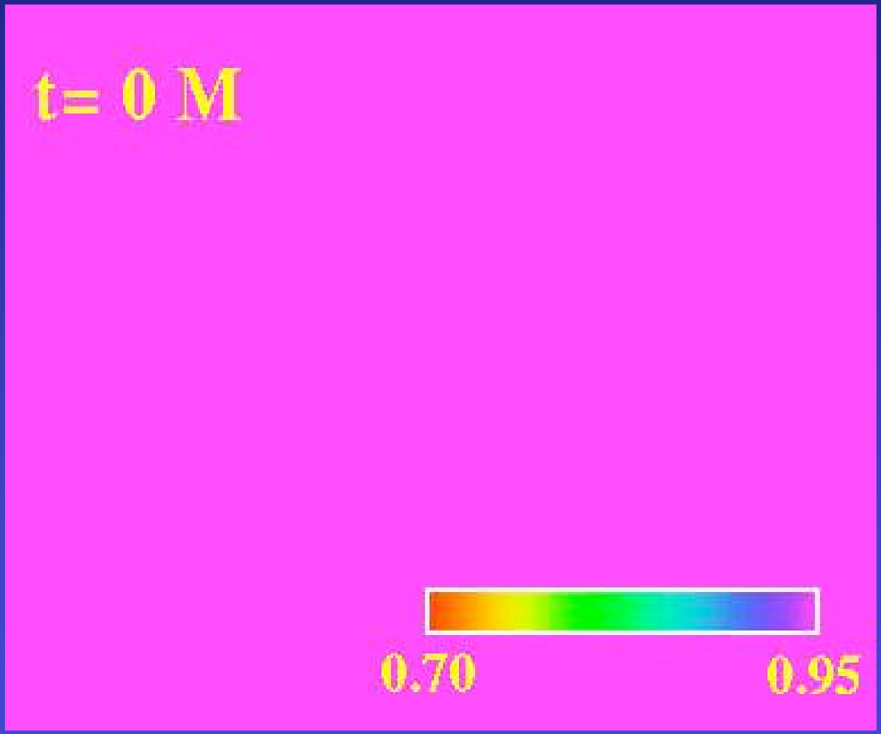


- two sample orbits from the *6/8h* resolution runs
- tuning *v* we are approaching the equivalent of a *homoclinic orbit*
- here the separation is close to 3M in the whirl part, which in the test-particle limit corresponds to the innermost stable circular *photon* orbit

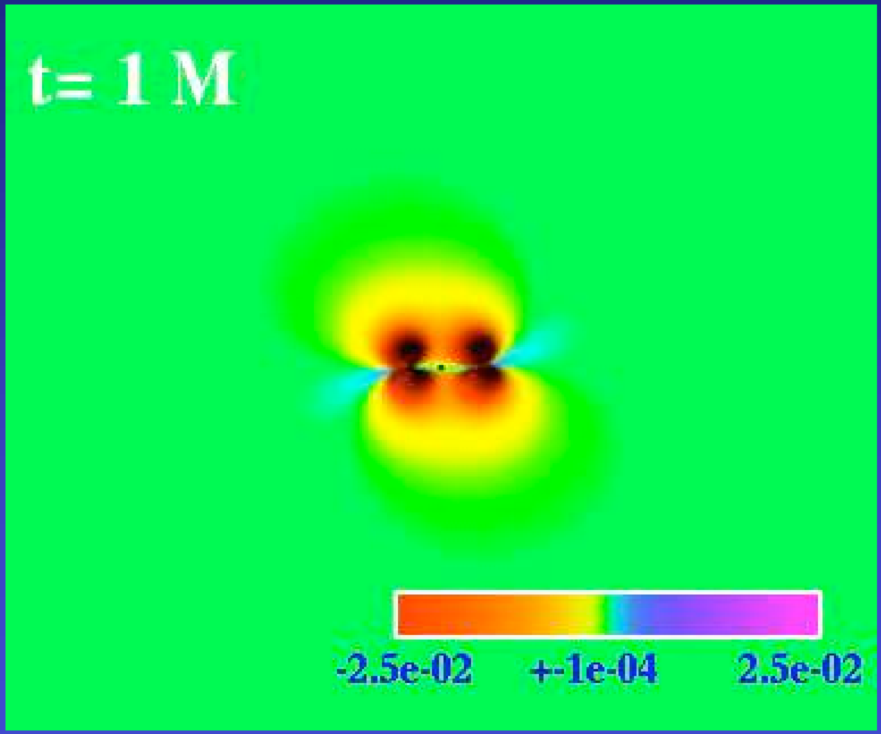
From N. Cornish and J. Levin, CQG 20, 1649 (2003)

Lapse and Gravitational Waves

6/8h resolution, v=0.21909 merger example

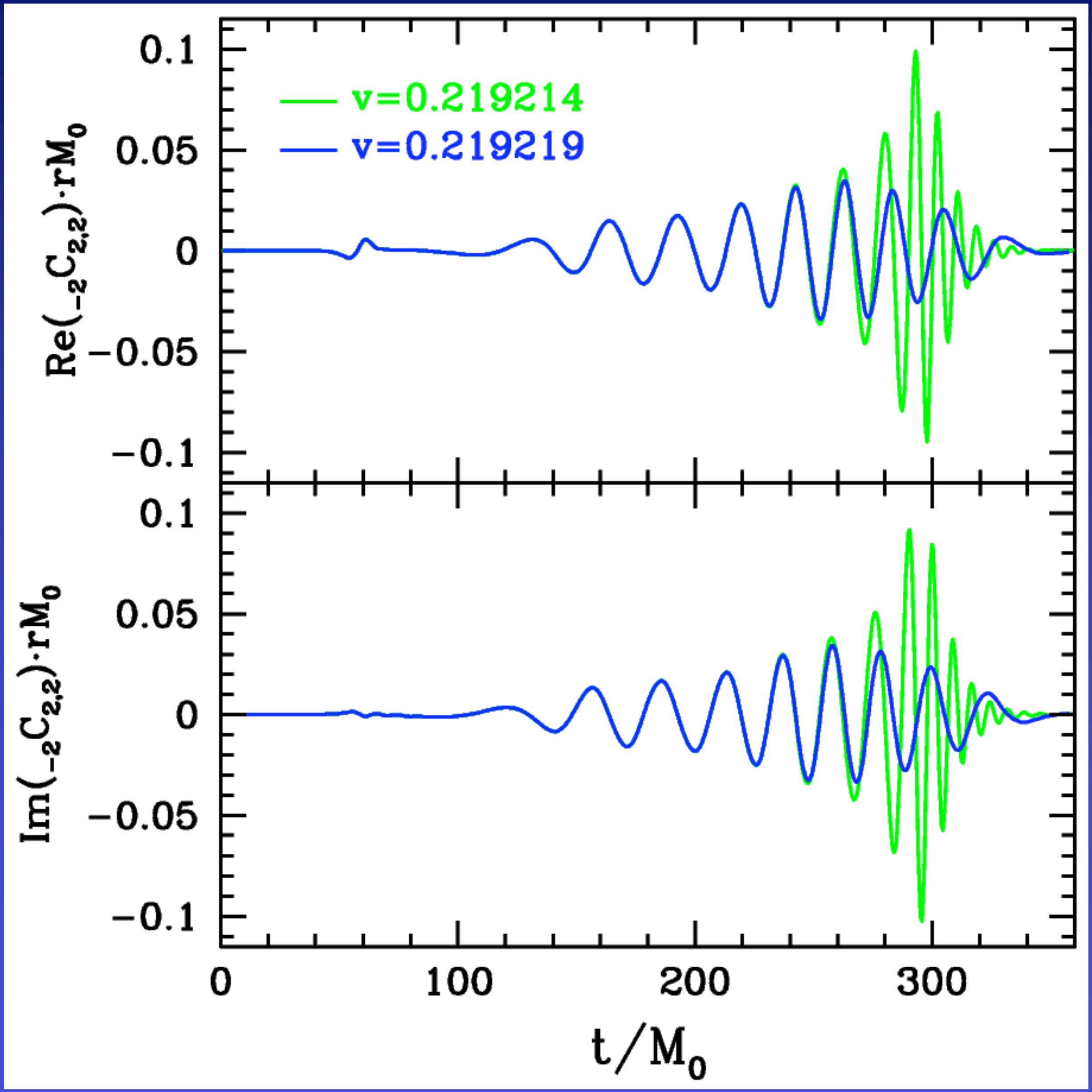


Lapse function α, orbital plane



Real component of the Newman-Penrose scalar Ψ_4(times rM), orbital plane

Waveforms



The real and imaginary components of the spin weight -2, l=2, m=2 spherical harmonic component of Ψ_4 times rM, measured at a coordinate distance of $50M$ from the center of the orbit, from the two $6/8\ h$ resolution simulations fine-tuned the most

How far can this go?

- System is losing energy, and quite rapidly, so there must be a limit to the number of orbits we can get
- Hawking's area theorem: assume cosmic censorship and "reasonable" forms of matter, then net area of all black holes in the universe can *not* decrease with time
  - the area of a single, isolated black hole is:

$$\boxed{A = 8\pi M^2\left(1+\sqrt{1-\frac{J^2}{M^4}}\right)}$$

  - initially, we have two non-rotating $(J=0)$ black holes, each with mass $M/2$:

$$\boxed{\sum A_i = 8\pi M^2}$$

  - maximum energy that can be extracted from the system is if the final black hole is also non-rotating:

$$\boxed{A_f = 16\pi M_f^2 \geq 8\pi M^2}$$

    in otherwords, the maximum energy that can be lost is a factor $1\text{-}1/\sqrt{2} \sim 29\%$

  - If the trend in the simulations continues, and the final $J{\sim}0.7M^2$, we still get close to 24% energy that could be radiated
    - the simulations further suggest around 1% energy is lost per whirl, so we may get as close to 20-30 orbits at the threshold of this fine-tuning process!

Summary and Outlook

- we are hopefully entering a very exciting time in astrophysics if the new gravitational wave detectors allow us to "see" the universe in gravitational waves for the first time
- we are also entering the era where numerical relativity will reveal the fascinating landscape of black hole coalescence
  - current simulations have only scratched the surface of binary configurations, whether of astrophysical or theoretical interest
    - first quasi-circular inspiral results are not "wild", but then again non-spinning, equal mass, zero-eccentricity orbits are about as plain as one can get
      - F. Herrmann, D. Shoemaker, P. Laguna *(gr-qc/0601026),* and Baker, Centrella, Choi, Koppitz, van Meter and Coleman Miller *(astro-ph/0603204)* studied black hole "kicks" from uequal mass mergers
      - Campanelli, Lousto and Zlochower *(gr-qc/0604012)* noted "orbital" hang-up in black holes with spins aligned with the orbital angular momentum of the binary
    - the tentative indications that zoom-whirl like behavior is present in the fully non-linear case hints that all of the interesting orbital behavior in test-particle orbits will also be present in the full problem